RANDOM WALKS AND CONTRACTING ELEMENTS I: DEVIATION INEQUALITY AND LIMIT LAWS

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Abstract. We study random walks on metric spaces with contracting elements. In this first article of the series, we establish the general framework for such random walks and establish sharp deviation inequalities using Gouëzel's pivotal time construction and Baik-Choi-Kim's pivoting technique. As a result, we obtain sharp rates of geodesic tracking of random walks on various spaces including CAT(0) spaces. We also revisit CLT and LIL on these spaces under optimal moment conditions.
Keywords. Random walk, CAT(0) space, Laws of large numbers, Central limit theorem, Geodesic tracking
MSC classes: 20F67, 30F60, 57M60, 60G50

Contents

1. Introduction

This is the first in a series of articles concerning random walks on metric spaces with contracting elements. This series is a reformulation of the previous preprint [Cho22a] announced by the author, aiming for clearer and more concise expositions. We note that many of the results in this series have been discussed in [Gou21], [BCK21], [Cho21a] and [Cho21b] for Gromov hyperbolic spaces and Teichmüller space. Still, we aim to present a more general theory in terms of contracting elements.

Convention 1.1. *Throughout, we assume that:*

- (X, d) *is a geodesic metric space;*
- G *is a countable group of isometries of* X, *and*

Date: January 2, 2023.

- *G contains two independent contracting isometries.*

We also fix a basepoint $o \in X$.

In a companion paper [Cho22c], we generalize this setting to also embrace asymmetric metric spaces that exhibit the bounded geodesic image property (BGIP). By doing so, we obtain a unified theory of random walks on the following spaces:

(1) (X, d) is a geodesic Gromov hyperbolic space and G contains two independent loxodromics [MT18], e.g. (X, d) is the curve complex of a finite-type hyperbolic surface and G is the corresponding mapping class group [MM99], or
(2) (X, d) is the complex of free factors of the free group of rank $N \geq 3$ and G is the outer automorphism group $\mathrm{Out}(F_N)$ [BF14];
(3) X is Teichmüller space of finite type, G is the corresponding mapping class group, and d is either the Teichmüller metric $d_{\mathcal{T}}$ or the Weil-Petersson metric d_{WP} ([Min96]; [Beh06], [BF09]);
(4) X is Culler-Vogtmann Outer space CV_N for $N \geq 2$, G is the outer automorphism group $Out(F_N)$, and d is the (asymmetric) Lipschitz metric d_{CV} [KMPT22];
(5) (X, d) is the Cayley graph of a braid group modulo its center $B_n/Z(B_n)$ with respect to its Garside generating set, and G is the braid group B_n [CW21];
(6) (X, d) is the Cayley graph of a group with nontrivial Floyd boundary [GP13];
(7) (X, d) is a (not necessarily proper nor finite-dimensional) CAT(0) space and G contains two independent strongly contracting isometries [BF09]; e.g., G is an irreducible right-angled Artin group and (X, d) is the universal cover of its Salvetti complex.

We emphasize that the spaces are not required to be proper nor separable in general. Note also that we do not assume that the action is properly discontinuous or WPD. The main results of this paper are as follows:

Theorem A (Deviation inequality)**.** *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G.*

(1) If μ has finite p-th moment for some $p > 0$, then there exists $K > 0$ such that for any $x \in X$ we have

$$\mathbb{E}\left[\sup_{n\geq 0} (\omega_n\, o, x)_o^p\right] < K, \quad \mathbb{E}\left[\sup_{n,n'\geq 0} (\check{\omega}_{n'} o, \omega_n\, o)_o^{2p}\right] < K. \tag{1.1}$$

(2) If μ has finite exponential moment, then there exist $\kappa, K > 0$ such that for any $x \in X$ we have

$$\mathbb{E}\left[\sup_{n\geq 0} e^{\kappa (x, \omega_n\, o)_o}\right] < K, \quad \mathbb{E}\left[\sup_{n,n'\geq 0} e^{\kappa(\check{\omega}_{n'} o, \omega_n\, o)_o}\right] < K. \tag{1.2}$$

This has been partially observed in Gromov hyperbolic spaces. Benoist and Quint observed the first item of Inequality 1.1 for cocompact actions [BQ16, Proposition 5.1]. Mathieu and Sisto observed the second item for acylindrical actions [MS20, Theorem 11.1]. Considering Benoist-Quint's examples in [BQ16, Subsection 5.5], these deviation inequalities are sharp.

Using deviation inequalities, we can obtain several limit laws for random walks. First, we observe the geodesic tracking of random walks with the optimal rate.

Theorem B (Geodesic tracking)**.** *Let (X, G, o) be as in Convention 1.1 and ω be the random walk generated by a non-elementary measure μ on G.*

(1) Suppose that μ has finite p-th moment for some $p > 0$. Then for almost every path $\omega = (\omega_n)_n$, there exists a quasigeodesic γ such that
$$\lim_n \frac{1}{n^{1/2p}} d(\omega_n\, o, \gamma) = 0.$$
(2) Suppose that μ has finite exponential moment. Then there exists $K < \infty$ satisfying the following: for almost every path $\omega = (\omega_n)_n$, there exists a quasigeodesic γ such that
$$\limsup_n \frac{1}{\log n} d(\omega_n\, o, \gamma) < K.$$

In particular, sublinear ($o(\sqrt{n})$, resp.) geodesic tracking occurs when the random walk has finite 1/2-th moment (finite first moment, resp.). We note that sublinear geodesic tracking under finite first moment assumption was established in Gromov hyperbolic spaces [Kai00], Teichmüller space ([Duc05], [Tio15]) and CAT(0) spaces [KM99], which was elaborated on by Gekhtman, Qing and Rafi [GQR22]. We also note Horbez's application of Benoist-Quint's deviation inequalities (with exponent $p-1$ instead of $2p$) to deduce the sublinear geodesic tracking of random walks with finite second moment [Hor18, Proposition 2.11].

In [Cho21a], the author obtained sharper geodesic tracking as in Theorem B for Gromov hyperbolic spaces and Teichmüller space. Nonetheless, the proof there required the endpoint stability of geodesics, i.e., geodesics with pairwise close endpoints fellow travel. We discuss another proof that does not rely on such a stability and only requires the presence of contracting elements.

Meanwhile, sublogarithmic tracking requires stronger moment conditions. Mathieu and Sisto established sublogarithmic tracking of random walks with finite exponential moments on acylindrically hyperbolic groups [MS20], and Maher and Tiozzo established the same result for random walks with finite support on weakly hyperbolic groups [MT18]; see also [Led01], [BHM11] and [Sis17] for related results.

We also deduce the central limit theorem (CLT) from the deviation inequality using Mathieu and Sisto's theory in [MS20]. Together with this, we obtain the law of the iterated logarithm (LIL) as follows.

Theorem C (CLT and LIL)**.** *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. If μ has finite second moment, then there exists a Gaussian law with variance $\sigma(\mu)^2$ to which $\frac{1}{\sqrt{n}}(d(o, \omega_n\, o) - n\lambda)$ converges in law. Here, λ is the escape rate of ω. Moreover, $\sigma(\mu) > 0$ if and only if μ is non-arithmetic. We also have*

$$\limsup_{n\to\infty} \pm \frac{d(o, \omega_n\, o) - \lambda n}{\sqrt{2n \log\log n}} = \sigma(\mu) \quad \textit{almost surely.}$$

We note that CLT has been discovered in Gromov hyperbolic spaces ([BQ16], [MS20]), Teichmüller space [Hor18], CAT(0) cube complices [FLM21] and CAT(0) spaces [LB22a]. Meanwhile, LIL has not been discussed except for Gromov hyperbolic spaces and Teichmüller space [Cho21a]. Moreover, the proof of the converse of CLT in [Cho21a] can also be generalized to the current setting; we will describe this in [Cho22b]. Finally, we also discuss a Berry-Esseen type estimates in Theorem 5.8.

1.1. **Other limit laws.** Our approach is based on Gouëzel's construction of pivotal times in [Gou21] and Baik-Choi-Kim's pivoting [BCK21]. By using the current theory, Gouëzel's results in [Gou21] can be discussed in the current general setting:

Theorem D (SLLN)**.** *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. Then there exists a constant $\lambda = \lambda(\omega) \in (0, +\infty]$ such that*

$$\lim_n \frac{1}{n} d(o, \omega_n\, o) = \lambda \tag{1.3}$$

for almost every ω. Moreover, $\lambda(\mu)$ is finite if and only if μ has finite first moment.

The SLLN for displacement is a consequence of the subadditive ergodic theorem, so the nontrivial part of the theorem is the strict positivity of the escape rate. For Gromov hyperbolic spaces, [MT18] proposed a very general framework and deduced SLLN with and without moment condition. On Teichmüller space, this is a consequence of the non-amenability of the mapping class group. Fernós, Lécureux and Mathéus also obtained the positivity on finite-dimensional CAT(0) cube complices under the finite first moment assumption. On general CAT(0) spaces, [KM99], [KL06] are relevant results. Recently, Le Bars used rank-1 isometries of proper CAT(0) spaces to deduce the convergence to the visual boundary and the SLLN for displacement [LB22b].

Theorem E (Exponential bound from below)**.** *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. Then for any $0 < L < \lambda(\mu)$, there exists $K > 0$ such that*

$$\mathbb{P}[d(o, \omega_n\, o) \le Ln] \le K e^{-n/K}$$

holds.

Even for free groups and Gromov hyperbolic spaces, this theorem has been observed very recently by Gouëzel. For other spaces that fall into Convention 1.1, this result seems new.

Corollary 1.2 (Large deviation principle). *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. If μ has finite exponential moment, then $\{d(o, \omega_n o)/n\}_n$ satisfies a large deviation principle with a proper convex rate function $I : [0, +\infty) \to [0, +\infty]$ which vanishes only at $\lambda = \lambda(\mu)$.*

Here, the LDP on Gromov hyperbolic spaces is due to Boulanger, Mathieu, Sert and Sisto [BMSS22]. They also provided a very general framework for LDP, namely, only requiring the space to possess a Schottky set. Since we provide Schottky sets for the setting of Convention 1.1, the existence of a proper convex rate function for LDP follows from Boulanger-Mathieu-Sert-Sisto's theory. Not only this, the usage of Schottky set in our theory is hugely influenced from their theory. The additional information furnished from Theorem E is that the rate function vanishes only at λ.

There is another dynamical quantity associated with a random isometry g, namely, its translation length $\tau(g)$. In [BCK21] and [Cho21a], the author and their coauthors proved the SLLN and the CLT for the translation length. These can be deduced from Corollary 5.6 proved here; we will discuss them in [Cho22b] using two different perspectives.

Our strategy is to bring Gouëzel's innovative idea of pivotal time construction [Gou21] and Baik-Choi-Kim's idea of pivoting for translation length [BCK21] into a more general setting. To achieve this, we first develop geometric lemmata that are preceded by many pioneers' observations. Since the main property we use is the contracting property, our theory applies to a wide range of spaces.

1.2. **Structure of the article.** In Section 2, we recall basic notions regarding contracting sets and random walks. In Section 3, we summarize the concatenation lemmata developed by Yang ([Yan14], [Yan19]) that were preceded by the observation of Bestvina and Fujiwara ([BF09]). We also provide a recipe to generate large enough Schottky sets out of two independent contracting isometries. In Section 4, we adapt Gouëzel's pivotal time construction and Baik-Choi-Kim's pivoting technique to the current setting, and observe the positivity of the escape rate. In Section 5, we establish deviation inequalities using the pivoting technique and discuss their consequences. In Section 6, we observe the geodesic tracking of random walks. In the appendix, we review Gouëzel's pivotal time construction using the concatenation lemmata for the sake of exposition.

Acknowledgments. The author thanks Hyungryul Baik, Talia Fernós, Ilya Gekhtman, Thomas Haettel, Joseph Maher, Hidetoshi Masai, Catherine Pfaff, Yulan Qing, Kasra Rafi, Samuel Taylor and Giulio Tiozzo for helpful discussions. The author is also grateful to the American Institute of Mathematics and the organizers and the participants of the workshop "Random walks beyond hyperbolic groups" in April 2022 for helpful and inspiring discussions.

The author was partially supported by Samsung Science & Technology Foundation grant No. SSTF-BA1702-01. This work constitutes a part of the author's PhD thesis.

2. Preliminaries

Throughout, we use the notation

$$(y,x)_z := \frac{1}{2}\big(d(y,x) + d(x,z) - d(y,z)\big)$$

for the *Gromov product* of y and z based at x. Given a path $\gamma : I \to X$, we denote by $\bar{\gamma}$ its reverse, i.e., $\bar{\gamma}(t) := \gamma(-t)$.

2.1. Contracting sets and bounded geodesic image property.

We introduce the notion of contracting sets. Intuitively, metric balls disjoint from these sets are seen as small.

Definition 2.1 (contracting sets). *For a subset $A \subseteq X$ of a metric space X and $\epsilon > 0$, we define the* closest point projection *of $x \in X$ to A by*

$$\pi_A(x) := \big\{a \in A : d_X(x,a) = d_X(x,A)\big\}.$$

A is said to be K-contracting *if:*

1. *$\pi_A(z) \neq \emptyset$ for all $z \in X$ and*
2. *for all $x, y \in X$ such that $d_X(x,y) \le d_X(x,A) - K$ we have*
$$\mathrm{diam}_X\left(\pi_A(x) \cup \pi_A(y)\right) \le K.$$

A K-contracting K-quasigeodesic is called a K-contracting axis.

Definition 2.2 (Bounded geodesic image property). *A subset $A \subseteq X$ of a geodesic metric space X is said to satisfy the K*-bounded geodesic image property*, or K-BGIP in short, if the following hold:*

1. *for any $z \in X$, $\pi_A(z) \neq \emptyset$;*
2. *for any geodesic η such that $\eta \cap \mathscr{N}_K(A) = \emptyset$, we have $\mathrm{diam}(\pi_A(\eta)) \le K$.*

A K-quasigeodesic that satisfies K-BGIP is called a K-BGIP axis.

We quote a lemma of Arzhantseva-Cashen-Tao.

Lemma 2.3 (Lemma 2.4, [ACT15]). *Let X be a geodesic space. Then a quasigeodesic in X is contracting if and only if it has BGIP.*

We now collect some properties of contracting axes.

Lemma 2.4 (Continuity of the projection). *For each $K > 1$ there exists a constant $K' = K'(K)$ that satisfies the following property.*

Let γ be a K-contracting axis and $x, y \in X$. Then $\pi_\gamma(\{x,y\})$ has diameter at most $K' + d(x,y)$.

Lemma 2.5 (Large projections are nearby)**.** *For each $K > 1$ there exists a constant $K' = K'(K)$ that satisfies the following property.*

Let $\gamma : I \to X$ be a K-contracting axis and $\eta : J \to X$ be a geodesic such that $\mathrm{diam}(\pi_\gamma(\eta)) > K'$*. Then for*

$$m := \inf \gamma^{-1}\pi_\gamma(\eta), \quad M := \sup \gamma^{-1}\pi_\gamma(\eta),$$

$\gamma([m, M] \cap I)$ is within Hausdorff distance K' from a subsegment of η.

Lemma 2.6 (Restrictions and nearby sets)**.** *For each $K > 1$ there exists a constant $K' = K'(K)$ such that any subsegment of a K-contracting axis is a K'-contracting axis.*

Moreover, if a set A is within Hausdorff distance K from a K-contracting axis and $\pi_A(z) \neq \emptyset$ for any $z \in X$, then A is K'-contracting.

Lemma 2.7 (No backtracking)**.** *For each $K > 1$ there exists a constant $K' = K'(K)$ that satisfies the following property.*

Let $\gamma : I \to X$ be a K-contracting axis, $\eta : J \to X$ be a geodesic and $\alpha_i \in J$ be such that $\alpha_1 \le \alpha_2 \le \alpha_3$. Let also $a_1, a_2, a_3 \in I$ be such that $\gamma(a_i) \in \pi_\gamma \eta(\alpha_i)$. Then a_1 and a_3 cannot both belong to $(-\infty, a_2 - K']$ nor $[a_2 + K', +\infty)$.

These are well-known to experts and we omit the proofs here. Nonetheless, interested readers can refer to the proofs in [Cho22c] in a more general setting that covers asymmetric metric spaces.

Definition 2.8 (Isometries with contracting properties)**.** *Let $K > 0$. An isometry g of X is said to be K-*contracting *if the orbit $n \in \mathbb{Z} \mapsto g^n o \in X$ is a K-contracting axis.*

Definition 2.9 (Translation length)**.** *For $g \in G$, the* (asymptotic) translation length *of g is defined by*

$$\tau(g) := \liminf_{n \to \infty} \frac{1}{n} d(o, g^n o).$$

An isometry has positive translation length if and only if its orbit $n \mapsto g^n o$ is a quasigeodesic.

Definition 2.10 ([BF09, Definition 5.8])**.** *Bi-infinite paths $\kappa = (x_i)_{i \in \mathbb{Z}}$, $\eta = (y_i)_{i \in \mathbb{Z}}$ are said to be* independent *if the map $(n, m) \mapsto d(x_n, y_m)$ is proper, i.e., for any $M > 0$, $\{(n, m) : d(x_n, y_m) < M\}$ is bounded.*

Isometries g, h of X are said to be independent *if their orbits are independent.*

Definition 2.11. *A subgroup of* $\mathrm{Isom}(X)$ *is said to be* non-elementary *if it contains two independent contracting isometries.*

Note that for $a, b \in \mathrm{Isom}(X)$ and $n, m \in \mathbb{Z} \setminus \{0\}$, a^n and b^m are independent contracting isometries if and only if a and b are so.

2.2. **Random walk.** Let μ be a probability measure on a discrete group G. We consider the *step space* $(G^{\mathbb{Z}}, \mu^{\mathbb{Z}})$, the product space of G equipped with the product measure of μ. Each element $(g_n)_n$ of the step space is called a *step path*, and there is a corresponding *sample path* $(\omega_n)_n$ under the correspondence

$$\omega_n = \begin{cases} g_1 \cdots g_n & n > 0 \\ id & n = 0 \\ g_0^{-1} \cdots g_{n+1}^{-1} & n < 0. \end{cases}$$

This structure constitutes a random walk with transition probability μ. We also introduce the notation $\check{g}_n = g_{-n+1}^{-1}$ and $\check{\omega}_n = \omega_{-n}$.

We define the *support* of μ, denoted by $\operatorname{supp}\mu$, as the set of elements in G that are assigned nonzero values of μ. $\langle \operatorname{supp}\mu \rangle$ and $\langle\langle \operatorname{supp}\mu \rangle\rangle$ denote the subgroup and the subsemigroup generated by the support of μ, respectively. In other words, we define

$$\begin{aligned} \langle \operatorname{supp}\mu \rangle &:= \{g_1 \cdots g_n : n \in \mathbb{Z}_{\geq 0}, g_i \in (\operatorname{supp}\mu) \cup (\operatorname{supp}\mu)^{-1}\}, \\ \langle\langle \operatorname{supp}\mu \rangle\rangle &:= \{g_1 \cdots g_n : n \in \mathbb{Z}_{\geq 0}, g_i \in \operatorname{supp}\mu\}. \end{aligned}$$

We denote by μ^N the product measure of N copies of μ, and by μ^{*N} the N-th convolution measure of μ. A measure μ is said to be *non-elementary* if $\langle\langle \operatorname{supp}\mu \rangle\rangle$ contains two independent contracting isometries. Note that by taking suitable powers if necessary, we may assume that two independent contracting isometries belong to the same $\operatorname{supp}\mu^{*N}$ for some $N > 0$. μ is said to be *non-arithmetic* if there exist $N > 0$ and $g, h \in \operatorname{supp}\mu^{*N}$ such that $\tau(g) \neq \tau(h)$. The random walk ω generated by μ is said to be admissible (non-elementary or non-arithmetic, resp.) if μ is admissible (non-elementary or non-arithmetic, resp.).

For each $p \geq 0$, we define the p-th moment of the probability measure μ on G by

$$\mathbb{E}_\mu[d(o, go)^p] := \int d(o, go)^p d\mu.$$

We also call it the p-th moment of the random walk ω generated by μ.

3. Concatenation of contracting axes

The goal of this section is to formulate and prove the following. Let $(\kappa_i)_i$ be a sequence of contracting axes that begin at x_i and terminate at y_i, respectively. Suppose that consecutive axes are well aligned: κ_i (κ_{i+1}, resp.) projects onto κ_{i+1} (κ_i, resp.) near x_{i+1} (y_i, resp.). Then we have a global alignment: κ_i projects onto κ_j near x_j or y_j, depending on whether $i < j$ or $j > i$.

We note that Proposition 3.6 and Lemma 3.8 were observed earlier in [Yan14, Section 3], and Lemma 3.7 follows from [Yan19, Proposition 2.9]. Nonetheless, we include their proofs as applications of Proposition 3.5.

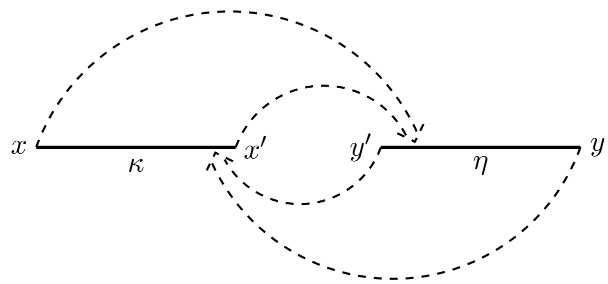

FIGURE 1. Schematics for an aligned sequence of paths.

Definition 3.1 (Witnessing). *Let $x, y, \{x_i\}_{i=1}^n, \{y_i\}_{i=1}^n$ be points in X and $D > 0$. We say that $[x, y]$ is* almost D-witnessed by $([x_1, y_1], \ldots, [x_n, y_n])$ *if the geodesic $[x, y]$ contains subsegments $[x_i', y_i']$ such that the following hold.*

(1) $d(x, y_i') \le d(x, x_{i+1}')$ for $i = 1, \ldots, n-1$, and
(2) $[x_i', y_i']$ and $[x_i, y_i]$ D-fellow travel for $i = 1, \ldots, n$.

Definition 3.2 (Alignment). *We say that a sequence (κ, η) of two paths κ, η is* aligned *if κ projects onto η near the beginning point of η and η projects onto κ near the terminating point of κ.*

More precisely, given paths κ from x to x' and η from y' to y, we say that (κ, η) is C-aligned if

$$\operatorname{diam}\big(x' \cup \pi_\kappa(\eta)\big) < C, \quad \operatorname{diam}\big(y' \cup \pi_\eta(\kappa)\big) < C.$$

In general, given paths κ_i from x_i to x_i' for each $i = 1, \ldots, n$, we say that $(\kappa_1, \ldots, \kappa_n)$ is C-aligned if

$$\operatorname{diam}\big(x_i' \cup \pi_{\kappa_i}(\kappa_{i+1})\big) < C, \quad \operatorname{diam}\big(x_{i+1} \cup \pi_{\kappa_{i+1}}(\kappa_i)\big) < C.$$

hold for $i = 1, \ldots, n-1$.

We can also put points in place of paths in the above definition; in that case, we regard points as degenerate paths that are endpoints of themselves. For example, given $y \in X$ and a path κ connecting x and x', we say that (κ, y) is C-aligned if $\operatorname{diam}(x' \cup \pi_\kappa(y)) < C$.

Note that if sequences $(\kappa_i, \ldots, \kappa_j)$ and $(\kappa_j, \ldots, \kappa_k)$ are C-aligned, then the sequence $(\kappa_i, \ldots, \kappa_j, \ldots, \kappa_k)$ is also C-aligned.

Our first lemma states that the projections of endpoints of two contracting axes onto each other govern the projections of the entire axes.

Lemma 3.3. *For each $C > 0$ and $K > 1$, there exists $D = D(K, C) > C$ that satisfies the following property.*

Let κ, η be K-contracting axes that connect x to x' and y to y', respectively. Suppose that (κ, y') and (x, η) are C-aligned. Then (κ, η) is D-aligned.

Note that one cannot expect similar consequences from the assumption that (κ, y') and (x', η) are aligned: imagine a long and thin isosceles triangle

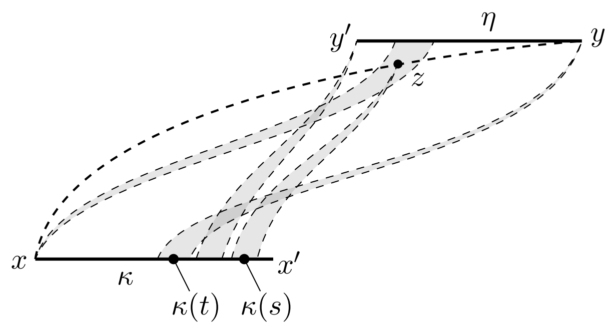

FIGURE 2. Schematics for Lemma 3.3

in the hyperbolic plane. Moreover, the assumption that (κ, y) and (x, η) are aligned also cannot guarantee the desired conclusion.

Proof. Recall Lemma 2.4: small sets are seen as small from a contracting axis. Hence, when κ is short, both $x' \cup \pi_\kappa(\eta) \subseteq \kappa$ and $y' \cup \pi_\eta(\kappa) \subseteq (y' \cup \pi_\eta(x)) \cup \pi_\eta(\kappa)$ are small; the same conclusion follows when η is short. Hence, it suffices to deal with the case that κ and η are long enough.

Note that if two endpoints of κ project onto η near y', then the entire κ also does so. This is due to the fact that $\pi_\kappa([x, x'])$ is coarsely between $\pi_\kappa(x)$ and $\pi_\kappa(x')$ (Lemma 2.7), and that $[x, x']$ and κ are nearby (Lemma 2.5). Similarly, if two endpoints of η project onto κ near x', then so does η. Hence, it suffices to prove that $y' \cup \pi_\eta(x')$ and $x' \cup \pi_\eta(y)$ are small.

To show the first item, suppose to the contrary that $\pi_\eta(x')$ is far from y'. Since $\pi_\eta(x)$ is close to y', the projection of $[x, x']$ onto η is large. Due to Lemma 2.5, $[x, x']$ passes through a small neighborhood of $\pi_\eta(x)$, which is near y'. Since κ and $[x, x']$ have small Hausdorff distance, we can take a point $p \in \kappa$ near y'. This implies that $d(y', \kappa)$ is small, and

$$d(y', x') \le d(y', \pi_\kappa(y')) + d(\pi_\kappa(y'), x')$$

is also small. This in turn means that

$$d(y', \pi_\eta(x')) \le d(y', x') + d(x', \eta)$$

is also small as desired.

For the next item, note that $\pi_\eta(x) \in \mathscr{N}_C(y')$ and $\pi_\eta(y) = y$ are distant since η is assumed to be long. Lemma 2.5 then tells us that $[x, y]$ passes through a neighborhood of y'; let $z \in [x, y]$ be the closest point to y'. Since y' projects onto κ near x', so does z. Then x, z, y are points on $[x, y]$ in order from left to right, and $\pi_\kappa(x)$ is near one endpoint x of κ while $\pi_\kappa(z)$ is near the other endpoint x'. By Lemma 2.7, $\pi_\kappa(y)$ is also near x' (rather than x) as desired. $\square$

In the previous lemma, the projection of y onto κ favored x' over x since $[x, y]$ had a large projection on η and passes through y'. We can put an arbitrary point p in place of y and expect the same phenomenon, given that the projection of p onto η does not favor y' over y. In other words, p either favors y' over y or favors x' over x. The following lemma captures this:

Lemma 3.4 (cf. [BF09, Lemma 5.6])**.** *For each $C > 0$ and $K > 1$, there exists $D = D(K, C) > C$ that satisfies the following property.*

Let κ, η be K-contracting axes that connect x to x' and y' to y, respectively. Suppose that (κ, η) is C-aligned. Then for any $p \in X$,

$$\mathrm{diam}(\pi_\eta(p) \cup y') \geq D \quad \textit{and} \quad \mathrm{diam}(\pi_\kappa(p) \cup x') \geq D$$

cannot happen simultaneously. Moreover, $\mathrm{diam}(\pi_\eta(p) \cup y') \geq D$ *implies* $d(p, \kappa) \geq d(p, \eta) + K$.

In other words, at least one of the following hold:

- (p, η) is D-aligned;
- (κ, p) is D-aligned.

Moreover, if the first item is not the case, then $d(p, \kappa) \geq d(p, \eta) + K$. Symmetrically, if the second item is not the case, then $d(p, \eta) \geq d(p, \kappa) + K$.

Proof. For the first assertion, let us put p in place of y in the final paragraph of the previous proof. If $\pi_\eta(p)$ and y' are far from each other, then $[x, p]$ has large projection on η and passes nearby y', say at a point z. The rest of the argument then tells us that $\pi_\kappa(z)$ and $\pi_\kappa(p)$ are near x'.

For the second assertion, suppose that (p, η) is not D-aligned, i.e., p projects onto η far from y'. Then $[p, y']$ passes through a neighborhood of $\pi_\eta(p)$ and we have

$$d(p, y') \geq d(p, \pi_\eta(p)) + d(\pi_\eta(p), y') - \mathrm{diam}(\pi_\eta(p)) - K_1 \geq d(p, \eta) + D/K^2 - 2K_1$$

for some suitable constant K_1.

Note also that since any point $q \in \kappa$ projects onto η near y', we deduce that $[p, q]$ passes through a neighborhood of y'. This implies that

$$d(p, q) \geq d(p, y') - K_1 \geq d(p, \eta) + K$$

for suitable constants K_1, D. Hence we have $d(p, \kappa) \geq d(p, \eta) + K$. □

We are now ready to prove the main result of this section.

Proposition 3.5. *For each $C > 0$ and $K > 1$, there exist $D = D(K, C) > C$ and $L = L(K, C) > C$ that satisfies the following.*

Let J be a nonempty set of consecutive integers, and $p, \{x_i, y_i\}_{i \in J}$ be points in X. For each $i \in J$, let κ_i be a K-contracting axis connecting x_i to y_i whose domain is longer than L. Suppose also that $(\kappa_i)_{i \in J}$ is C-aligned. Then we have the following:

(1) the statements

$$(\kappa_i, p) \textit{ is } D\textit{-aligned}, \quad (p, \kappa_i) \textit{ is } D\textit{-aligned}$$

cannot hold simultaneously;

(2) the set

$$J_0 = J_0\Big(p; (\kappa_i)_{i \in J}, D\Big) := \left\{ j \in J : \begin{array}{l} (\kappa_i, p) \text{ is } D\text{-aligned for } i \in J \text{ such that } i < j, \\ (p, \kappa_i) \text{ is } D\text{-aligned for } i \in J \text{ such that } i > j \end{array} \right\}$$

consists of either a single integer or two consecutive integers;

(3) $\pi_{\cup_i \kappa_i}(p)$ *is nonempty and is contained in* $\bigcup\{\pi_{\kappa_j}(p) : j \in J_0\}$*; and*

(4) (κ_l, κ_m) *is* D*-aligned for any* $l, m \in J$ *such that* $l < m$.

Proof. Let $D = D(K, C)$ be as in Lemma 3.3 and 3.4. For the first item, we take large enough L such that $\mathrm{diam}(x_i \cup \pi_{\kappa_i}(p)) < D$ and $\mathrm{diam}(y_i \cup \pi_{\kappa_i}(p)) < D$ cannot happen simultaneously. For example, $L = K(2D + 2K)$ will do. This choice will guarantee the following for each $i \in J$:

$$\begin{aligned} &\mathrm{diam}(x_i \cup \pi_{\kappa_i}(p)) < D \Rightarrow \mathrm{diam}(y_i \cup \pi_{\kappa_i}(p)) \geq D, \\ &\mathrm{diam}(y_i \cup \pi_{\kappa_i}(p)) < D \Rightarrow \mathrm{diam}(x_i \cup \pi_{\kappa_i}(p)) \geq D. \end{aligned} \tag{3.1}$$

This implies that J_0 cannot contain two elements of J that are separated by more than 1. Hence, it suffices to show that J_0 is nonempty.

Suppose, say, there exists m such that (κ_i, p) is D-aligned for all integer $i \geq m$ (in particular, J is not bounded above). Then Inequality 3.1 says that (p, κ_i) is not D-aligned for $i \geq m$, and Lemma 3.4 asserts that $d(p, \kappa_{m+n}) < d(p, \kappa_m) - nK$ for all $n \geq 0$; this violates the nonnegativity of the metric. Hence, such m cannot exist and

$$\{i \in J : (\kappa_i, p) \text{ is } D\text{-aligned}\}$$

cannot contain an infinite increasing sequence of consecutive integers. In other words, J is bounded above unless

$$S := \{j \in J : (\kappa_j, p) \text{ is not } D\text{-aligned}\}$$

is nonempty. If S is empty and J is bounded above, then $\max J \in J_0$ clearly holds. Now suppose that S is nonempty and let $j \in S$. Then (κ_j, p) is not D-aligned, which implies that (p, κ_{j+1}) is D-aligned and (κ_{j+1}, p) is not D-aligned if $j + 1 \in J$. The induction goes on: (p, κ_i) is D-aligned and (κ_i, p) is not D-aligned for all $i \in J$ such that $i > j$. $(*)$ Note also that for any $k \leq \inf S$, (κ_i, p) is D-aligned for all $i \in J$ such that $i < k$. This implies that $\min S \in J_0$ if S has the minimum.

The remaining case is that S is nonempty but S does not have the minimum: that means, both J and S is not bounded below. In this case, $(*)$ implies that (κ_i, p) is not D-aligned for all $i \in J$. By Lemma 3.4, we then have $d(p, \kappa_i) < d(p, \kappa_j) - K(j - i)$ for all $i, j \in J$ such that $i < j$. Fixing j and taking small enough i, we obtain a contradiction with the nonnegativity of the metric. Hence, this case does not happen and the second item is established.

We now prove the third and the fourth items. First suppose that J_0 is a singleton $\{j\}$. By definition and Inequality 3.1, we have:
(3.2)
$$\begin{aligned} &\operatorname{diam}(x_i \cup \pi_{\kappa_i}(p)) < D, \quad \operatorname{diam}(y_i \cup \pi_{\kappa_i}(p)) > D \quad (i \in J \text{ such that } i > j),\\ &\operatorname{diam}(y_i \cup \pi_{\kappa_i}(p)) < D, \quad \operatorname{diam}(x_i \cup \pi_{\kappa_i}(p)) > D \quad (i \in J \text{ such that } i < j). \end{aligned}$$

If $\operatorname{diam}(y_j \cup \pi_{\kappa_j}(p)) < D$ holds in addition, then $j+1$ also belongs to J_0, a contradiction. Hence, we have $\operatorname{diam}(y_i \cup \pi_{\kappa_i}(p)) \geq D$ for $i \in J$ such that $i \geq j$. Then Lemma 3.4 tells us that $d(y, \kappa_{i+1}) > d(y, \kappa_i)$ for $i \in J \setminus \sup J$ such that $i \geq j$. By a similar reason, $d(y, \kappa_{i-1}) > d(y, \kappa_i)$ for $i \in J \setminus \inf J$ such that $i \leq j$. Hence we conclude $\pi_{\cup_i \kappa_i}(p) = \pi_{\kappa_j}(p)$.

When $J_0 = \{j, j+1\}$, we similarly deduce $\pi_{\cup_i \kappa_i}(p) \subseteq \pi_{\kappa_j}(p) \cup \pi_{\kappa_{j+1}}(p)$.

Let us now take $l, m \in J$ such that $l < m$. We want to show that (κ_l, κ_m) is D-aligned, or equivalently, $\operatorname{diam}(y_l \cup \pi_{\kappa_l}(p)) < D$ for any $p \in \kappa_m$ and $\operatorname{diam}(x_m \cup \pi_{\kappa_m}(p)) < D$ for any $p \in \kappa_l$. Both directly follow from the assumption if $l = m-1$. When $l < m-1$, $J_0 = J_0(p)$ for $p \in \kappa_m$ must contain m because of the third item. Then the second item implies that $l < J_0(p)$ and $\operatorname{diam}(y_l \cup \pi_{\kappa_l}(p)) < D$ as desired. Similarly, $p \in \kappa_l$ implies $J_0(p) < m$ and $\operatorname{diam}(x_m \cup \pi_{\kappa_m}(p)) < D$ as desired. □

We just proved that $d(p, \kappa_{i+1}) > d(p, \kappa_i)$ for $i \in J \setminus \sup J$ such that $i \in J_0$. Here, we are using Inequality 3.2 for $i \in J \setminus \sup J$ only; in particular, this holds even if $\kappa_{\sup J}$ is short.

Proposition 3.6. *For each $C > 0$ and $K > 1$, there exist $E = E(K, C) > C$ and $L = L(K, C) > C$ that satisfy the following. Let $x, y \in X$ and $\kappa_1, \ldots, \kappa_N$ be K-BGIP axes whose domains are longer than L.*

If $(x, \kappa_1, \ldots, \kappa_N, y)$ is C-aligned, then (x, κ_i, y) is E-witnessed for each $i = 1, \ldots, N$. Moreover, $p \in \mathscr{N}_E([x, y])$ and $(x, y)_p < E$ for any $p \in \kappa_i$.

Proof. Proposition 3.5 and Lemma 2.5 guarantee that the following statements hold for suitable choices of E_1, E_2 and L.

First, (x, κ_1) is E_1-aligned and hence (κ_1, x) is not E_1-aligned. This prevents $J_0\big(x; (\kappa_i)_i, E_1\big)$ from containing elements larger than 1, i.e., $J_0\big(x; (\kappa_i)_i, E_1\big) = \{1\}$. By a similar reason, we have $J_0\big(y; (\kappa_i)_i, E_1\big) = \{N\}$. Consequently we have that (x, κ_i, y) is E_1-aligned for each $i = 1, \ldots, N$. Since κ_i is a long enough K-BGIP axis, there exists a subsegment $[x', y']$ of $[x, y]$ that is within Hausdorff distance E_2 from κ_i. □

We next discuss the contracting of the concatenation of an aligned sequence of contracting axes.

Lemma 3.7. *For each $C, M > 0$ and $K > 1$, there exist $K' = K'(K, C, M) > C$ and $L = L(K, C) > C$ that satisfies the following.*

Let J be a nonempty set of consecutive integers and $\{x_i, y_i\}_{i \in J}$ be points in X. For each $i \in J$, let κ_i be a K-contracting axis connecting x_i and y_i whose domain is longer than L. Suppose that $(\kappa_i)_{i \in J}$ is C-aligned and $d(y_i, x_{i+1}) < M$ for $i \in J \setminus \sup J$. Then $\cup_i \kappa_i$ is a K'-contracting axis.

Proof. There exist large enough $K_1, E, L > K$ such that the following argument works.

To show that $\cup_i \kappa_i$ is contracting, pick $x, y \in X$. If $\mathrm{diam}(\pi_{\kappa_i}(x) \cup \pi_{\kappa_i}(y)) > K_1$ for some i, then $[x, y]$ passes through the K_1-neighborhood of κ_i. If not, i.e., if the projections of x and y onto each κ_i are close to each other, we claim that their projections onto $\cup_i \kappa_i$ are also close to each other.

Let $D = D(K, C) > K$ be as in Proposition 3.5 and let $j \in J_0 = J_0(x; (\kappa_i)_i, D)$. Then we have the following cases:

(1) $\pi_{\kappa_j}(x)$ is distant from both x_j and y_j: then so is $\pi_{\kappa_j}(y)$, and it follows that $J_0(y; (\kappa_i)_i, D) = \{j\}$ also. Hence the projections of x and y onto $\cup_i \kappa_i$ are those onto κ_j, which are close to each other.

(2) $\pi_{\kappa_j}(x)$ is close to x_j: then $J_0(x; (\kappa_i)_i, D) \subseteq \{j-1, j\}$, and $\pi_{\kappa_i}(x)$ is far from x_i for $i \neq j$. Since $\pi_{\kappa_i}(x)$ and $\pi_{\kappa_i}(y)$ are close to each other, the same conclusion holds for $\pi_{\kappa_i}(y)$'s. In other words, $J_0(y; (\kappa_i)_i, D) \subseteq \{j-1, j\}$. In this case,

$$\begin{aligned}\mathrm{diam}\left(\pi_{\cup_i \kappa_i}(\{x, y\})\right) &\le \mathrm{diam}\left(\pi_{\kappa_j}(\{x, y\}) \cup \pi_{\kappa_{j-1}}(\{x, y\})\right) \\ &\le \mathrm{diam}\left(\pi_{\kappa_j}(\{x, y\}) \cup x_j\right) + \mathrm{diam}(x_j \cup y_{j-1}) \\ &+ \mathrm{diam}\left(y_{j-1} \cup \pi_{\kappa_{j-1}}(x)\right) + \mathrm{diam}\left(\pi_{\kappa_{j-1}}(\{x, y\})\right)\end{aligned}$$

is bounded. Here, the first and the last term are bounded thanks to the assumption. The second term is at most M, and the third term is also bounded since $j \in J_0(x; (\kappa_i)_i, D)$ so (κ_{j-1}, x) is D-aligned.

(3) $\pi_{\kappa_i}(x)$ is close to x_{j+1}: an argument similar to the one above works.

We now show that $\cup_i \kappa_i$ is a quasigeodesic. Note that for any $i < j < k$ and $x \in \kappa_i$, $y \in \kappa_j$ and $z \in \kappa_k$, then $(x, \kappa_{i+1}, \ldots, \kappa_j, \ldots, \kappa_{k-1}, z)$ is C-aligned and $(x, z)_y < E$ due to Proposition 3.6. In fact, $(x, z)_y$ is also when $x \in \kappa_i$, $z \in \kappa_{i+1}$ and $y = x_{i+1}$. Indeed, (x, κ', z) is C-aligned for the restriction κ' of κ_{i+1} between y and z, so Proposition 3.6 tells us that $(x, z)_y < E$ if $d(y, z) > E$; if not $(x, z)_y \le d(y, z)$ is clearly bounded by E.

These bounds on the Gromov products imply the following. For $i < j$, $x \in \kappa_i$ and $y \in \kappa_j$, we have

$$\begin{aligned}d(x, y) &\ge d(x, y_i) + d(x_{i+1}, y_{i+1}) + \ldots + d(x_j, y) - |j - i|E \\ &\ge \frac{1}{2}[d(x, y_i) + d(x_{i+1}, y_{i+1}) + \ldots + d(x_j, y)] - E.\end{aligned}$$

Here, we used the fact that $d(x_k, y_k) \ge \frac{L}{K} - K \ge 2E$ for each k. Since each κ_i is a K-quasigeodesic, we can conclude that $\cup_i \kappa_i$ is also a quasigeodesic. $\square$

The latter part of the previous proof still works even when $d(y_i, x_{i+1})$ is not uniformly bounded, given that the intermediate segments are included. Hence, we obtain the following:

Lemma 3.8. *For each $C > 0$ and $K > 1$, there exist $K' = K'(K, C) > C$ and $L = L(K, C) > C$ that satisfy the following.*

Let J be a nonempty set of consecutive integers and $\{x_i, y_i\}_{i \in J}$ be points in X. For each $i \in J$, let κ_i be a K-contracting axis connecting x_i and y_i whose domains are longer than L. Suppose that $(\kappa_i)_{i \in J}$ is C-aligned. Then the concatenation Γ of $(\ldots, [x_{i-1}, y_{i-1}], [y_{i-1}, x_i], [x_i, y_i], [y_i, x_{i+1}], \ldots)$ is a K'-quasigeodesic.

3.1. **Schottky set.** Using the previous concatenation lemma, we can construct arbitrarily many independent directions out of two independent contracting isometries.

Lemma 3.9. *Let $K > 1$ and $\kappa = (x_i)_{i \in \mathbb{Z}}$, $\eta = (y_i)_{i \in \mathbb{Z}}$ be independent K-contracting axes. Then κ projects onto η small. More precisely, there exists $K' > 0$ such that*

$$\operatorname{diam}(x_0 \cup \pi_\kappa(\eta)) < K'.$$

Moreover, the projection of the forward half of γ onto its backward half is also small. More precisely, K' can be chosen so that

$$\operatorname{diam}\left(x_0 \cup \pi_{\{x_i\}_{i \geq 0}}\left(\{x_i\}_{i \leq 0}\right)\right) < K'.$$

Proof. Let $K_1 = K'(K)$ be as in Lemma 2.5. Let $l \in \mathbb{Z}$ be such that $x_l \in \pi_\kappa(y_0)$. For the first assertion, suppose to the contrary and let $n_i, m_i \in \mathbb{Z}$ be such that $|m_i| \geq i$ and $x_{m_i} \in \pi_\kappa(y_{n_i})$. Note that $|n_i|$ escapes to infinity, as $\cup_{|k| \leq M} \pi_\kappa(y_k)$ is finite for each M. Moreover, since κ, η are K-quasigeodesics, we have $d(x_l, x_{m_i}), d(y_0, y_{n_i}) > K$ for large enough i. For those i's, Lemma 2.5 implies that x_{m_i} is contained in the K_1-neighborhood of $[y_0, y_{n_i}]$, which is contained in the K_1-neighborhood of η. In particular, we have $d(x_{m_i}, y_{n_i'}) < 2K_1$ for some $n_i' \in \mathbb{Z}$. This contradicts the independence of κ and η, and we are led to the conclusion.

The second assertion can be deduced in a similar way, using the fact that the forward and the backward half-paths diverge from each other. $\square$

In practice, we employ the restrictions of κ and η on various sets J of consecutive integers. This necessitates the following modification.

Lemma 3.10. *Let $K > 1$ and $\kappa = (x_i)_{i \in \mathbb{Z}}$, $\eta = (y_i)_{i \in \mathbb{Z}}$ be independent K-contracting axes. Then there exists $K' > 0$ such that the following hold:*

(1) $\kappa|_J := (x_i)_{i \in J}$, $\eta|_J := (y_i)_{i \in J}$ are K'-contracting axes for any set J of consecutive integers;
(2) for any set J of consecutive integers that contains 0, we have

$$\operatorname{diam}\left(x_0 \cup \pi_{\kappa|_J}(\eta)\right) < K';$$

(3) for any positive integer M we have

$$\operatorname{diam}\left(x_0 \cup \pi_{\{x_0, \ldots, x_M\}}(\{x_i : i \leq 0\})\right) < K'$$

Proof. The first item is a part of Lemma 2.6; let $K_1 = K'(K)$ be as in Lemma 2.6 and $K_2 = K'(K_1)$ be as in Lemma 2.5. Let also $l \in \mathbb{Z}$ be such that $y_l \in \pi_\eta(x_0)$ and let $d(x_0, y_l) = D$.

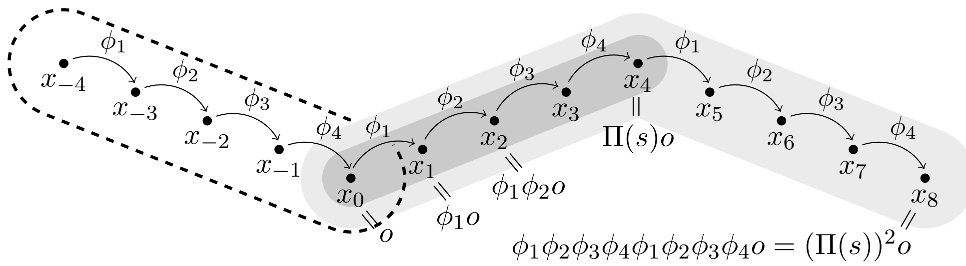

FIGURE 3. Axes associated with a sequence of isometries $s = (\phi_1, \phi_2, \phi_3, \phi_4)$. Points inside the darker shadow constitute $\Gamma(s)$, and those inside the lighter shadow constitute $\Gamma^2(s)$. Points inside the dashed region constitute $\Gamma^{-1}(s)$.

The proof for the second item is almost identical to the proof of the previous lemma. First suppose to the contrary; we take $n_i, m_i \in \mathbb{Z}$ and sets of consecutive integers J_i containing 0 such that $|m_i| \geq i$ and x_{m_i} belongs to the projection of y_{n_i} onto $\kappa|_{J_i}$. Again, for each M we have

$$\begin{aligned}\bigcup_{|i|\leq M,\ 0\in J\subseteq \mathbb{Z}} \pi_{\{x_j : j\in J\}}(y_i) &\subseteq \bigcup_{|i|\leq M} \{x_j : d(y_i, x_j) \leq d(y_i, x_0)\} \\ &\subseteq \bigcup_{|i|\leq M} \{x_j : d(x_0, x_j) \leq d(x_0, y_i)\} \\ &\subseteq \{x_j : d(x_0, x_j) \leq D + 2KM + 2K\},\end{aligned}$$

which is a finite set. Hence, $|n_i|$ necessarily escapes to infinity. Moreover, since κ, η are K-quasigeodesics, we have $d(x_0, x_{m_i}), d(y_l, y_{n_i}) > K$ for large enough i. Moreover, $\kappa|_{J_i}$ have the K_1-contracting for all i. Lemma 2.5 then asserts that x_{m_i} is within the K_2-neighborhood of $[x_0, y_{n_i}]$, since it has large projection on $\pi|_{J_i}$. Moreover, it is contained in the $(2K_2 + D)$-neighborhood of η. We thus have $d(x_{m_i}, y_{n_i'}) < 3K_2 + D$ for some n_i', which contradicts the independence of κ and η. Hence we are led to the conclusion. Similar trick works for the third item. $\square$

We will now construct a path out of a sequence of isometries by applying them to the reference point o. Given a sequence $s = (\phi_i)_{i=1}^k$ of isometries of X, we denote the product of its entries $\phi_1 \cdots \phi_k$ by $\Pi(s)$. Now let

$$x_{nk+i} := \Pi(s)^n \phi_1 \cdots \phi_i o = (\phi_1 \cdots \phi_k)^n \phi_1 \cdots \phi_i o$$

for each $n \in \mathbb{Z}$ and $i = 0, \ldots, k-1$. We let $\Gamma^m(s) := (x_0, x_1, \ldots, x_{mk})$ when $m \geq 0$ and $\Gamma^m(s) := (x_0, x_{-1}, \ldots, x_{mk})$ when $m < 0$. When $m = 1$, we usually omit the superscript and write $\Gamma(s) = (x_0, \ldots, x_k)$. Finally, let $\Gamma^{\pm\infty}(s) = (x_i)_{i\in\mathbb{Z}}$. Note that $\Gamma^m(s)$ is a concatenation of $|m|$ translates of $\Gamma(s)$ or its reverse.

Definition 3.11 (cf. [Gou21, Definition 3.11])**.** *Let $K > 0$ and $S \subseteq G^M$ be a set of sequences of M isometries. We say that S is K-*Schottky *if the following hold:*

1. *$\Gamma^m(s)$ is a K-contracting axis for all $s \in S$ and $m \in \mathbb{Z}$;*
2. *for each $x \in X$, all element $s \in S$ except at most 1 satisfies that $(x, \Gamma^n(s))$ is K-aligned for all $n \in \mathbb{Z}$;*
3. *for each $x \in X$ and $s \in S$, if $(x, \Gamma^n(s))$ is not K-aligned for some $n > 0$ ($n < 0$, resp.) then $(x, \Gamma^m(s))$ is K-aligned for all $m \le 0$ ($m \ge 0$, resp.).*

An intuitive example of a Schottky set is the set S of all words of length n in $F_2 = \langle a, b\rangle$ that consists of letters a and b (not involving a^{-1} and b^{-1}). For any infinite ray on F_2, there exists at most 1 element $s \in S$ that matches the direction. Moreover, s and s^{-1} diverge early for any $s \in S$. Note also that the set of the self-concatenations of these words also satisfy the same property. This means that we can make the directions made by two words in S to diverge early (compared to their lengths). This model will help understanding the following proposition.

Proposition 3.12 (cf. [Gou21, Proposition 3.12])**.** *For any $N_0 > 0$, there exists a K-Schottky set of cardinality N_0 in $(\operatorname{supp}\mu)^m$ for some m and K.*

Proof. Since μ is a non-elementary measure, there exist independent BGIP isometries $a, b \in \langle\langle \operatorname{supp}\mu\rangle\rangle$. By taking suitable powers if necessary, we may assume that $a = \Pi(\alpha)$, $b = \Pi(\beta)$ for some sequences $\alpha, \beta \in (\operatorname{supp}\mu)^N$ for some N. Then $\Gamma^{\pm\infty}(\alpha)$, $\Gamma^{\pm\infty}(\beta)$ are independent contracting axes.

Let:

- $K_1 = K'$ be as in Lemma 3.10 for $\Gamma^{\pm\infty}(\alpha), \Gamma^{\pm\infty}(\beta)$;
- $K_2 = D(K_1)$, $L_2 = L'(K_1)$ be as in Proposition 3.5;
- $K_3 = K'(K_1)$, $L_3 = L'(K_1)$ be as in Lemma 3.7.

Note here that $\Gamma^{\pm\infty}(\alpha)$, $\Gamma^{\pm\infty}(\beta)$ are unchanged after replacing α, β with their self-concatenations. Hence, by self-concatenating α and β if necessary, we may assume that $N > \max(L_2, L_3)$. This choice forces the following: for any $x \in X$, the statements

$$\big(x, \Gamma(\alpha)\big) \text{ is } K_2\text{-aligned}, \quad \big(\Gamma(\alpha), x\big) \text{ is } K_2\text{-aligned}$$

are mutually exclusive. Analogous statements for β are also mutually exclusive. Let us now pick an integer M such that $2^M > N_0$. Since any subset of a Schottky set is again Schottky, we aim to make a Schottky set of cardinality 2^M.

We will consider the set S' of sequences of MN isometries that are concatenations of α's and β's, i.e.,

$$S' := \left\{(\phi_i)_{i=1}^{MN} \in G^{MN} : (\phi_{N(k-1)+1}, \ldots, \phi_{Nk}) \in \{\alpha, \beta\} \text{ for } k = 1, \ldots, M\right\}.$$

Given $s = (\phi_i)_{i=1}^{MN} \in S'$, we have defined

$$x_{nMN+i}(s) = (\phi_1 \cdots \phi_{MN})^n \phi_1 \cdots \phi_i o$$

for $n \in \mathbb{Z}$ and $i = 0, \ldots, MN-1$. We temporarily define sub-axes of the main axis $\Gamma(s)$, namely,

$$\Gamma_k(s) := \left(x_{N(k-1)}(s), \ldots, x_{Nk}(s)\right),$$
$$\Gamma_{-k}(s) := \left(x_{-N(k-1)}(s), \ldots, x_{-Nk}(s)\right)$$

for $k = 1, \ldots, M$. Then for each m, $\Gamma^m(s)$ is a concatenation of $\Gamma_k(s)$'s and their translates, which are translates of $\Gamma(\alpha)$ and $\Gamma(\beta)$. These translates are K_1-contracting axes whose domains are longer than L_2. Moreover, Lemma 3.10 tells us that $\left(\bar{\Gamma}(\gamma), \Gamma(\gamma')\right)$ is K_1-aligned for $\gamma, \gamma' \in \{\alpha, \beta\}$. Lemma 3.7 then implies that $\Gamma^m(s)$ is a K_3-contracting axis.

We now fix $x \in X$. Let us first consider the condition:

$$\left(x, \Gamma_M(s)\right) \text{ is } K_2\text{-aligned.} \tag{3.3}$$

We claim that if $s \in S'$ satisfies this condition, then $\pi_{\Gamma^n(s)}(x)$'s are uniformly bounded for $n \geq 0$. For each n, note that $\Gamma^n(s)$ is a concatenation of K_1-contracting axes

$$(\kappa_i)_{i=1}^{MN} = \left(\Gamma_1(s),\ \ldots,\ \Gamma_M(s),\ \Pi(s)\Gamma_1(s),\ \ldots,\ \Pi(s)\Gamma_M(s),\ \ldots,\ \Pi(s)^{n-1}\Gamma_M(s)\right).$$

Each pair of consecutive axes are of the form $\left(g\bar{\Gamma}^{-1}(\gamma), g\Gamma(\gamma')\right)$ for some $g \in G$ and $\gamma, \gamma' \in \{\alpha, \beta\}$, which is K_1-aligned due to Lemma 3.10. Hence, we can apply Proposition 3.5. Note that Condition 3.3 implies that $\left(\Gamma_M(s), x\right)$ is not K_2-aligned. This means that $J_0 = J_0\left(x; (\kappa_i)_i, K_2\right)$ and $\{M{+}1, \ldots, MN\}$ are disjoint. Therefore, $\pi_{\Gamma^n(s)}(x)$ is contained in $\Gamma_1(s) \cup \ldots \cup \Gamma_M(s) = \Gamma(s)$ and

$$\operatorname{diam}\left(\pi_{\Gamma^n(s)}(x) \cup o\right) \leq \operatorname{diam}(\Gamma(s)) \leq K_3 MN + K_3.$$

By a similar reason, the condition

$$\left(x, \Gamma_{-M}(s)\right) \text{ is } K_2\text{-aligned} \tag{3.4}$$

implies $\operatorname{diam}\left(\pi_{\Gamma^n(s)}(x) \cup o\right) \leq \operatorname{diam}\left(\Gamma^{-1}(s)\right) \leq K_3 MN + K_3$ for all $n \leq 0$. These can be summarized as follows.

Observation 3.13. *If $s \in S'$ satisfy Condition 3.3 and 3.4, then*

$$\operatorname{diam}\left(\pi_{\Gamma^n(s)}(x) \cup o\right) < K_3 MN + K_3$$

holds for all $n \in \mathbb{Z}$.

We now consider the case that an element of S' violates these conditions.

Observation 3.14. *If $s = (\phi_i)_{i=1}^{MN} \in S'$ violates Condition 3.3, then all the other elements $s' = (\phi'_i)_{i=1}^{MN} \in S'$ satisfy Condition 3.3.*

To show this, let k be the first index such that $(\phi_{N(k-1)+1}, \ldots, \phi_{Nk})$ and $(\phi'_{N(k-1)+1}, \ldots, \phi'_{Nk})$ differ. By switching the roles of α and β if necessary, we may assume that

$$(\phi_{N(k-1)+1}, \ldots, \phi_{Nk}) = \alpha, \quad (\phi'_{N(k-1)+1}, \ldots, \phi'_{Nk}) = \beta.$$

Let us denote $x_i(s)$ by x_i and $x_i(s')$ by x'_i.

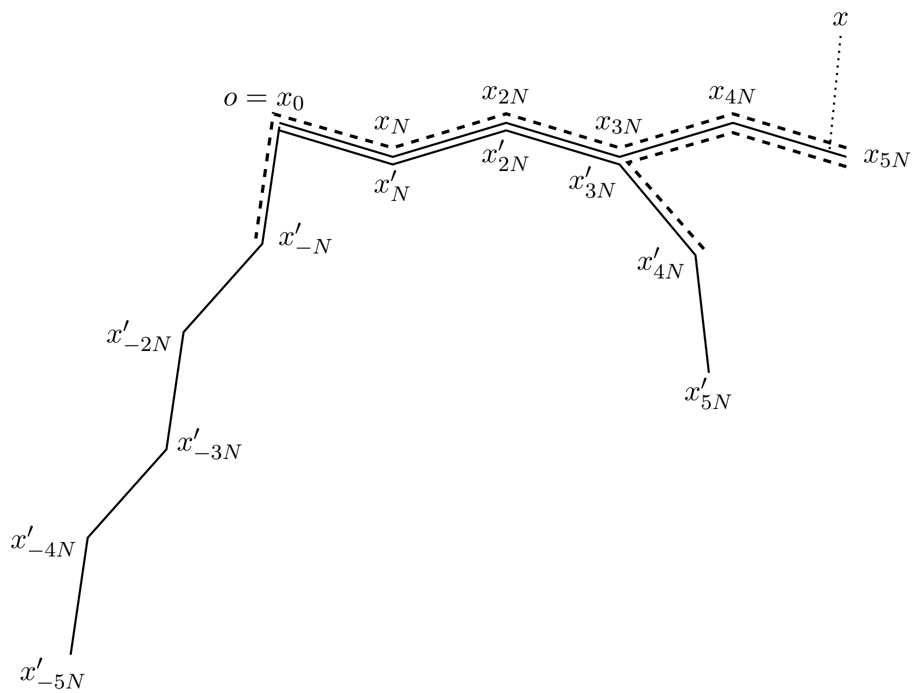

FIGURE 4. Schematics for Lemma 3.12. Three solid lines represent $\Gamma(s)$, $\Gamma(s')$ and $\Gamma^{-1}(s')$ in the clockwise order. The upper dashed line represents the concatenation of $\bar{\Gamma}_M(s), \ldots, \bar{\Gamma}_1(s)$ and $\bar{\Gamma}_{-1}(s')$. The lower dashed line represents the concatenation of $\bar{\Gamma}_M, \ldots, \bar{\Gamma}_k(s)$ and $\Gamma_k(s')$.

Note that the path

$$\Big(x_{MN},\ x_{MN-1},\ \ldots,\ x_{(k-1)N} = x'_{(k-1)N},\ x'_{(k-1)N+1},\ \ldots,\ x'_{kN}\Big)$$

is the concatenation of K_1-contracting axes

$$(\eta_i)_{i=1}^{M-k+2} := \left(\bar{\Gamma}_M(s), \bar{\Gamma}_{M-1}(s), \ldots, \bar{\Gamma}_k(s), \Gamma_k(s')\right).$$

(See the lower dashed line in Figure 4.) Each pair of consecutive axes are of the form $\left(g\bar{\Gamma}^{-1}(\gamma), g\Gamma(\gamma')\right)$ for some $\gamma, \gamma' \in \{\alpha, \beta, \alpha^{-1}, \beta^{-1}\}$ such that $\gamma \neq \gamma'$. Lemma 3.10 implies that such pair is K_1-aligned, which allows us to apply Proposition 3.5.

In particular, since we are assuming that $\left(\bar{\Gamma}_M(s), x\right)$ is not K_2-aligned, $J_0 = J_0(x; (\eta_i)_i, K_2)$ is necessarily $\{1\}$ and $(x, \eta_{M-k+2}) = (x, \Gamma_k(s'))$ is K_2-aligned. We then apply Proposition 3.5 to $\Gamma^n(s')$, a concatenation of K_1-contracting axes

$$(\kappa'_i)_{i=1}^{MN} = \left(\Gamma_1(s'),\ \ldots,\ \Gamma_M(s'),\ \Pi(s')\Gamma_1(s'),\ \ldots,\ \Pi(s')\Gamma_M(s'),\ \ldots,\ \Pi(s')^{n-1}\Gamma_M(s')\right).$$

Then $J_0' = J_0(x; (\kappa_i')_i, K_2)$ and $\{k+1, \ldots, MN\}$ are disjoint, which implies Condition 3.3 for s' and

$$\pi_{\Gamma^n(s')}(x) \in \Gamma_1(s') \cup \cdots \cup \Gamma_k(s') \subseteq \Gamma(s'),$$
$$\operatorname{diam}\left(\pi_{\Gamma^n(s')}(x) \cup o\right) \le \operatorname{diam}(\Gamma(s')) \le K_3MN + K_3$$

for all $n \ge 0$.

A similar argument leads to the following.

Observation 3.15. *If $s \in S'$ violates Condition 3.4, then all the other elements in S' satisfy Condition 3.4.*

Our next claim concerns the third item.

Observation 3.16. *If $s = (\phi_i)_{i=1}^{MN} \in S'$ violates Condition 3.3, then all elements $s' = (\phi_i')_{i=1}^{MN} \in S'$ (including $s' = s$) satisfy Condition 3.4.*

To show this, observe that the path

$$\left(x_{MN},\ x_{MN-1},\ \ldots,\ x_0 = o,\ x'_{-1},\ \ldots,\ x'_{-N}\right)$$

is the concatenation of K_0-contracting axes $\bar{\Gamma}_M(s)$, $\ldots$, $\bar{\Gamma}_1(s)$ and $\bar{\Gamma}_{-1}(s')$. (See the upper dashed line in Figure 4.) This sequence is again K_1-aligned, even in the case $s = s'$, by Lemma 3.10. As before, we can apply Proposition 3.5 and deduce that $\pi_{\Gamma_{-1}(s')}(x) \cup o$ has diameter less than K_2. Now Proposition 3.5 in turn implies

$$\pi_{\Gamma^{-n}(s')}(x) \in \Gamma_{-1}(s'),\ \operatorname{diam}(\pi_{\Gamma^{-n}(s')}(x) \cup o) \le \operatorname{diam}(\Gamma_{-1}(s')) \le K_3N + K_3$$

for all $n \ge 0$.

An analogous statement follows.

Observation 3.17. *If $s = (\phi_i)_{i=1}^{MN} \in S'$ violates Condition 3.4, then all elements $s' = (\phi_i')_{i=1}^{MN} \in S'$ (including $s' = s$) satisfy Condition 3.3.*

Let us summarize the observations and finish the proof. We take $K = K_3MN + K_3$. The first item was established before. The second item is equivalent to saying that both Condition 3.3 and Condition 3.4 are satisfied by all but at most 1 element of S'. The third item is equivalent to saying that Condition 3.3, 3.4 cannot be violated at the same time by any element of S'. We have the following 4 cases.

- Every $s \in S'$ satisfies Condition 3.3 and Condition 3.4: then clearly the second and the third items hold.
- Some $s \in S'$ violates Condition 3.3: then Condition 3.3 is satisfied by all the other elements of S' and Condition 3.4 is satisfied by all elements of S':
- Some $s \in S'$ violates Condition 3.4: then Condition 3.4 is satisfied by all the other elements of S' and Condition 3.3 is satisfied by all elements of S'.
- Some $s \in S'$ simultaneously violates Condition 3.3 and 3.4; this case is ruled out by the previous 2 cases.

In all cases, we conclude that the second and the third items hold. □

The following property is immediate.

Lemma 3.18. *Let S be a K-Schottky set in G^m for $m > 2K^2$. Then for any $s, s' \in S$, we have*

$$\operatorname{diam}\left(\pi_{\Gamma^{-1}(s')}(\Pi(s)o) \cup o\right) < K, \quad \operatorname{diam}\left(\pi_{\Gamma(s)}(\Pi(s')^{-1}o) \cup o\right) < K. \tag{3.5}$$

Proof. For the first inequality, we observe that

$$\operatorname{diam}\left(\pi_{\Gamma(s)}(\Pi(s)o) \cup o\right) = \operatorname{diam}\left(\Pi(s)o \cup o\right) \geq m/K - K > K.$$

Hence, we observe that

$$\operatorname{diam}\left(\pi_{\Gamma^n(s')}(\Pi(s)o) \cup o\right) \leq K$$

holds for all n if $s \neq s'$ (Property (2)), and for $n \leq 0$ if $s = s'$ (Property (3)); hence the first inequality.

We can analogously deduce the second inequality. □

We will use Schottky sets to guarantee alignments. In order to fully utilize the previous alignment lemmata, it is important to prepare Schottky sets whose elements have sufficiently long domains.

From now on we fix an integer $N_0 > 410$. Let $K_0 := K(N_0)$ be as in Proposition 3.12, and

- $K_1 := K'(K_0)$ be as in Lemma 2.4,
- $K_2 := K'(K_0)$ be as in Lemma 2.5,
- $K_3 := K'(K_0)$ be as in Lemma 2.7,
- $D_0 := D(K_0, K_0 + K_1 + K_2 + K_3)$ be as in Lemma 3.3 and 3.4;
- for $i = 1, 2$, $D_i := D(K_0, D_{i-1})$, $L_i := L(K_0, D_{i-1})$ be as in Lemma 3.3, 3.4 and Proposition 3.5;
- $E_0 := E(K_0, D_2)$, $L_3 := L(K_0, D_2)$ be as in Proposition 3.6.

Let us now fix a K_0-Schottky set $S \subseteq (\operatorname{supp}\mu)^{M_0}$ of cardinality at least N_0. Note that the n-self-concatenations of elements of S also comprise a K_0-Schottky set. Hence, we may assume that

$$M_0 > L_1 + L_2 + L_3 + 20K_0(K_0 + E_0). \tag{3.6}$$

From now on, K_0-contracting axes of the form $\Gamma^m(s)$ for $s \in S$ and $m \neq 0$ are called *Schottky axes.*

4. Pivotal times and pivoting

4.1. **Pivotal times.**

We adapt Gouëzel's pivotal time construction in [Gou21] to our setting. Most of the proofs will be deferred to Appendix A, since the original lemmata are already proved in [Gou21]; see also [Cho21a].

Let $(w_i)_{i=0}^{\infty}$, $(v_i)_{i=1}^{\infty}$ be isometries in G. Now given a sequence

$$s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_n, \beta_n, \gamma_n, \delta_n) \in S^{4n},$$

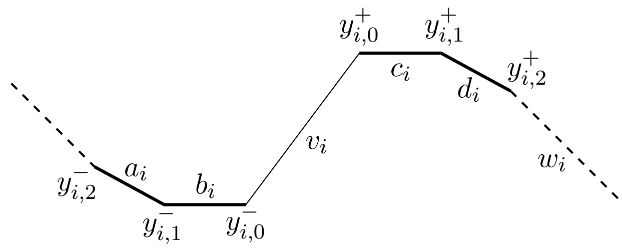

FIGURE 5. $y^{\pm}_{i,k}$ inside a trajectory.

we first define

$$a_i := \Pi(\alpha_i),\ b_i := \Pi(\beta_i)\ c_i := \Pi(\gamma_i),\ d_i := \Pi(\delta_i). \tag{4.1}$$

We then consider isometries that are subwords of

$$w_0 a_1 b_1 v_1 c_1 d_1 w_1 \cdots a_k b_k v_k c_k d_k w_k \cdots .$$

More precisely, we set the initial case $w^+_{-1,2} := id$ and define

$$\begin{aligned} w^-_{i,2} &:= w^+_{i-1,2} w_{i-1}, & w^-_{i,1} &:= w^-_{i,2} a_i, & w^-_{i,0} &:= w^-_{i,2} a_i b_i, \\ w^+_{i,0} &:= w^-_{i,2} a_i b_i v_i, & w^+_{i,1} &:= w^-_{i,2} a_i b_i v_i c_i, & w^+_{i,2} &:= w^-_{i,2} a_i b_i v_i c_i d_i \end{aligned}$$

and the translates $y^{\pm}_{i,t} = w^{\pm}_{i,t} o$ of o by them. We also employ notations

$$\begin{aligned} \Upsilon(\alpha_i) &:= w^-_{i,2}\Gamma(\alpha_i), & \Upsilon(\beta_i) &:= w^-_{i,1}\Gamma(\beta_i), \\ \Upsilon(\gamma_i) &:= w^+_{i,0}\Gamma(\gamma_i), & \Upsilon(\delta_i) &:= w^+_{i,1}\Gamma(\delta_i). \end{aligned}$$

for simplicity. We will later consider modified versions of a given sequence s such as $\tilde{s} = (\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i, \tilde{\delta}_i)_{i=1}^n$ or $\bar{s} = (\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i, \bar{\delta}_i)_{i=1}^n$. We also employ notations analogous to the above for these choices, i.e., $\tilde{a}_i$, …, $\tilde{d}_i$, $\bar{a}_i$, …, $\bar{d}_i$, $\tilde{w}^{\pm}_{i,j}$, $\bar{w}^{\pm}_{i,j}$ and $\Upsilon(\tilde{\alpha}_i)$, …, $\Upsilon(\tilde{\delta}_i)$, $\Upsilon(\bar{\alpha}_i)$, …, $\Upsilon(\bar{\delta}_i)$.

We now define the *set of pivotal times* and $P_n = P_n\left(s, (w_i)_{i=0}^n, (v_i)_{i=1}^n\right)$ and an auxiliary moving point $z_n = z_n\left(s, (w_i)_{i=0}^n, (v_i)_{i=1}^n\right)$ inductively. First set $P_0 = \emptyset$ and $z_0 = o$. Now given $P_{n-1} \subseteq \{1, \ldots, n-1\}$ and $z_{n-1} \in X$, P_n and z_n are determined as follows.

(A) When $(z_{n-1}, \Upsilon(\alpha_n))$, $\left(\Upsilon(\beta_n), y^+_{n,1}\right)$, $\left(y^-_{n,0}, \Upsilon(\gamma_n)\right)$ and $\left(\Upsilon(\delta_n), y^-_{n+1,2}\right)$ are K_0-aligned, then we set $P_n = P_{n-1} \cup \{n\}$ and $z_n = y^+_{n,1}$.

(B) Otherwise, we seek sequences $\{i(1) < \cdots < i(N)\} \subseteq P_{n-1}$ $(N > 1)$ such that

$$\left(\Upsilon(\delta_{i(1)}), \Upsilon(\alpha_{i(2)}), \Upsilon(\beta_{i(2)}), \ldots, \Upsilon(\alpha_{i(N)}), \Upsilon(\beta_{i(N)})\right)$$

is D_0-aligned and $\left(\Upsilon(\beta_{i(N)}), y^-_{n+1,2}\right)$ is K_0-aligned.

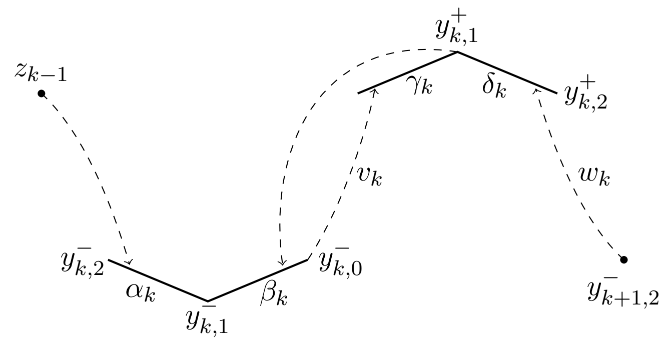

FIGURE 6. Schematics for Criteria 4.2, 4.3, 4.4 and 4.5.

If exists, let $\{i(1) < \cdots < i(N)\}$ be such a sequence with maximal $i(1)$; we set $P_n = P_{n-1} \cap \{1, \ldots, i(1)\}$ and $z_n = y^-_{i(N),1}$. If such a sequence does not exist, then we set $P_n = \emptyset$ and $z_n = o$.[1]

One reason for defining P_n is that it records the Schottky axes aligned along $[o, \omega_n o]$. More precisely, we have:

Lemma 4.1. *Let $P_n = \{i(1) < \ldots < i(m)\}$. Then*

$$\left(o, \Upsilon(\alpha_{i(1)}), \Upsilon(\beta_{i(1)}), \Upsilon(\gamma_{i(1)}), \Upsilon(\delta_{i(1)}), \ldots, \Upsilon(\alpha_{i(m)}), \Upsilon(\beta_{i(m)}), \Upsilon(\gamma_{i(m)}), \Upsilon(\delta_{i(m)}), y^-_{n+1,2}\right)$$

is a subsequence of a D_0-aligned sequence of Schottky axes. In particular, it is D_1-aligned.

This is originally from [Gou21, Lemma 5.3] and we defer the proof to Appendix A.

From now on, let us endow the Schottky set S with the uniform measure and consider the product measure on S^{4n}. In other words, we assume that $\alpha_i, \beta_i, \gamma_i, \delta_i$ are drawn from S independently. We now discuss when new pivotal time is added to the set of pivotal times; this tells us how to pivot the direction a pivotal time without affecting the set of pivotal times.

Lemma 4.2. *For $1 \le k \le n$, $s \in S^{4(k-1)}$, we have*

$$\mathbb{P}\left(\#P_k(s, \alpha_k, \beta_k, \gamma_k, \delta_k) = \#P_{k-1}(s) + 1\right) \ge 1 - 4/N_0.$$

Proof. Recall Criterion (A) for $\#P_k = \#P_{k-1} + 1$. We will investigate four required conditions one-by-one.

First, the condition

$$\operatorname{diam}\left(\pi_{\Upsilon(\gamma_k)}(y^-_{k,0}) \cup y^+_{k,0}\right) = \operatorname{diam}\left(\pi_{\Gamma(\gamma_k)}(v_k^{-1} o) \cup o\right) < K_0 \tag{4.2}$$

depends only on γ_k. This holds for at least $(\#S - 1)$ choices in S.

[1] When there are several sequences that realize maximal $i(1)$, we choose the maximum in the lexicographic order on the length of sequences and $i(2)$, $i(3)$, ...

Similarly, the condition

$$\operatorname{diam}\left(\pi_{\Upsilon(\delta_k)}(y^-_{k+1,2}) \cup y^+_{k,2}\right) = \operatorname{diam}\left(\pi_{\Gamma^{-1}(\delta_k)}(w_k o) \cup o\right) < K_0 \tag{4.3}$$

depends only on δ_k, and holds for at least $(\#S-1)$ choices in S.

Now fixing the choice of γ_k, the condition

$$\operatorname{diam}\left(\pi_{\Upsilon(\beta_k)}(y^+_{k,1}) \cup y^-_{k,0}\right) = \operatorname{diam}\left(\pi_{\Gamma^{-1}(\beta_k)}(v_k c_k o) \cup o\right) < K_0 \tag{4.4}$$

depends only on β_k. This holds for at least $(\#S-1)$ choices in S.

This time, let us fix the choice of $s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}, \delta_{k-1})$; in particular, $w^-_{k,2}$ and z_{k-1} are now determined. Then the condition

$$\operatorname{diam}\left(\pi_{\Upsilon(\alpha_k)}(z_{k-1}) \cup y^-_{k,2}\right) = \operatorname{diam}\left(\pi_{\Gamma(\alpha_k)}\left((w^-_{k,2})^{-1} z_{k-1}\right) \cup o\right) < K_0 \tag{4.5}$$

depends on α_k. This holds for at least $(\#S-1)$ choices of α_k.

In summary, the probability that Criterion (A) holds is at least

$$\frac{\#S-1}{\#S} \cdot \frac{\#S-1}{\#S} \cdot \frac{\#S-1}{\#S} \cdot \frac{\#S-1}{\#S} \geq 1 - \frac{4}{N_0} \qquad \square$$

Given α_1, β_1, γ_1, δ_1, …, α_{k-1}, β_{k-1}, $\gamma_{k-1}, \delta_{k-1}$, we define the set $\tilde{S}_k$ of triples $(\alpha_k, \beta_k, \gamma_k)$ in S^3 that satisfy Condition 4.2, 4.4 and 4.5.

Note that $\tilde{S}_k$ takes up large portion of S^3: in the previous proof we observed that $\#\left[S^3 \setminus \tilde{S}_k\right] \leq 3(\#S)^2$. Moreover, for $(\alpha_k, \beta_k, \gamma_k) \in \tilde{S}_k$, $\{(\alpha_k, \beta'_k, \gamma_k) \in \tilde{S}_k : \beta_k \in S\}$ has at least $\#S-1$ elements. In addition, $\tilde{S}_k$ is the set of allowed choices for pivoting:

Lemma 4.3. *Let $i \in P_k(s)$ for a choice $s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_n, \beta_n, \gamma_n, \delta_n)$, and $\bar{s}$ be obtained from s by replacing $(\alpha_i, \beta_i, \gamma_i)$ with*

$$(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i) \in \tilde{S}_i(\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}, \delta_{i-1}).$$

Then $P_l(s) = P_l(\bar{s})$ and $\tilde{S}_l(s) = \tilde{S}_l(\bar{s})$ for any $1 \leq l \leq k$.

This corresponds to [Gou21, Lemma 5.7], whose proof can be found in Appendix A.

Given $1 \leq k \leq n$ and a partial choice $s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_k, \beta_k, \gamma_k, \delta_k)$, we say that $\bar{s} = (\bar{\alpha}_1, \bar{\beta}_1, \bar{\gamma}_1, \bar{\delta}_1, \ldots, \bar{\alpha}_k, \bar{\beta}_k, \bar{\gamma}_k, \bar{\delta}_k)$ is *pivoted from s* if:

- $\delta_j = \bar{\delta}_j$ for all $1 \leq j \leq k$,
- $(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i) \in \tilde{S}_i(s)$ for each $i \in P_k(s)$, and
- $(\alpha_j, \beta_j, \gamma_j) = (\bar{\alpha}_j, \bar{\beta}_j, \bar{\gamma}_j)$ for each $j \in \{1, \ldots, k\} \setminus P_k(s)$.

Lemma 4.3 then asserts that being pivoted from each other is an equivalence relation. For each $s \in S^{4k}$, let $\mathcal{E}_k(s)$ be the equivalence class of s. Our central estimation follows:

Lemma 4.4. *For $1 \leq k \leq n$, $j \geq 0$ and $s \in S^{4(k-1)}$, the probability*

$$\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - j \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s),\ (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right)$$

is less than $(4/N_0)^{j+1}$.

This corresponds to [Gou21, Lemma 5.8] and we defer the proof to Appendix A.

Corollary 4.5. *When* $s = (\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^n$ *is chosen from* S^{4n} *with the uniform measure,* $\#P_n(s)$ *is greater in distribution than the sum of* n *i.i.d.* X_i*, whose distribution is given by*

$$\mathbb{P}(X_i = j) = \begin{cases} (N_0 - 4)/N_0 & \text{if } j = 1, \\ (N_0 - 4)4^{-j}/N_0^{-j+1} & \text{if } j < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

More generally, the distribution of $\#P_{k+n}(s) - \#P_k(s)$ *conditioned on the choices of* $(\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^k$ *also dominates the sum of* n *i.i.d.* X_i.

Moreover, we have $\mathbb{P}(\#P_n(s) \le (1 - 10/N_0)n) \le e^{-Kn}$ *for some* $K > 0$.

This corresponds to [Gou21, Lemma 5.9, Proposition 5.10].

4.2. **Pivoting for pairs of independent paths.** In this subsection, we deal with two independent paths at the same time. Let us fix two K_0-Schottky sets S, $\check{S}$ of cardinality N_0. We then fix isometries $(w_j)_{j=0}^\infty$, $(v_j)_{j=1}^\infty$, $(\check{w}_j)_{j=0}^\infty$ and $(\check{v}_j)_{j=1}^\infty$. We also draw choices $s = (\alpha_j, \beta_j, \gamma_j, \delta_j)_{j=1}^n \in S^{4n}$ and $\check{s} = (\check{\alpha}_j, \check{\beta}_j, \check{\gamma}_j, \check{\delta}_j)_{j=1}^n \in \check{S}^{4n}$.

We first construct the set of pivotal times on the words

$$\begin{aligned} w &= w_0 a_1 b_1 v_1 c_1 d_1 \cdots a_n b_n v_n c_n d_n w_n, \\ \check{w} &= \check{w}_0 \check{a}_1 \check{b}_1 \check{v}_1 \check{c}_1 \check{d}_1 \cdots \check{a}_n \check{b}_n \check{v}_n \check{c}_n \check{d}_n \check{w}_n \end{aligned}$$

separately. Let $\mathcal{E}$, $\check{\mathcal{E}}$ be equivalence classes made by the pivoting for w and $\check{\omega}$, respectively. Let also

$$P(\mathcal{E}) = \{i(1) < i(2) < \ldots\}, \quad P(\check{\mathcal{E}}) = \{\check{i}(1) < \check{i}(2) < \ldots\}.$$

We will now construct

$$\begin{aligned} S_1^*(s, \check{s}) &:= S_1^*, \\ \check{S}_1^*(s, \check{s}) &:= \check{S}_1^*(\alpha_{\check{i}(1)}), \\ S_2^*(s, \check{s}) &:= \check{S}_2^*(\check{\alpha}_{i(1)}, \check{\beta}_{\check{i}(1)}, \check{\gamma}_{\check{i}(1)}, \alpha_{i(1)}, \beta_{i(1)}, \gamma_{i(1)}), \\ \check{S}_2^*(s, \check{s}) &:= \check{S}_2^*(\check{\alpha}_{\check{i}(1)}, \check{\beta}_{\check{i}(1)}, \check{\gamma}_{\check{i}(1)}, \alpha_{i(1)}, \beta_{i(1)}, \gamma_{i(1)}, \alpha_{i(2)}), \\ &\vdots \end{aligned}$$

for $1 \le k \le M$. We first consider

$$\phi_k := (\check{w}_{\check{i}(k),2}^-)^{-1} w_{i(k),2}^- = \check{w}_{\check{i}(k)}^{-1} \check{d}_{\check{i}(k)-1}^{-1} \check{c}_{\check{i}(k)-1}^{-1} \cdots \check{w}_0^{-1} \cdot w_0 a_1 b_1 v_1 c_1 d_1 \ldots w_{i(k)}.$$

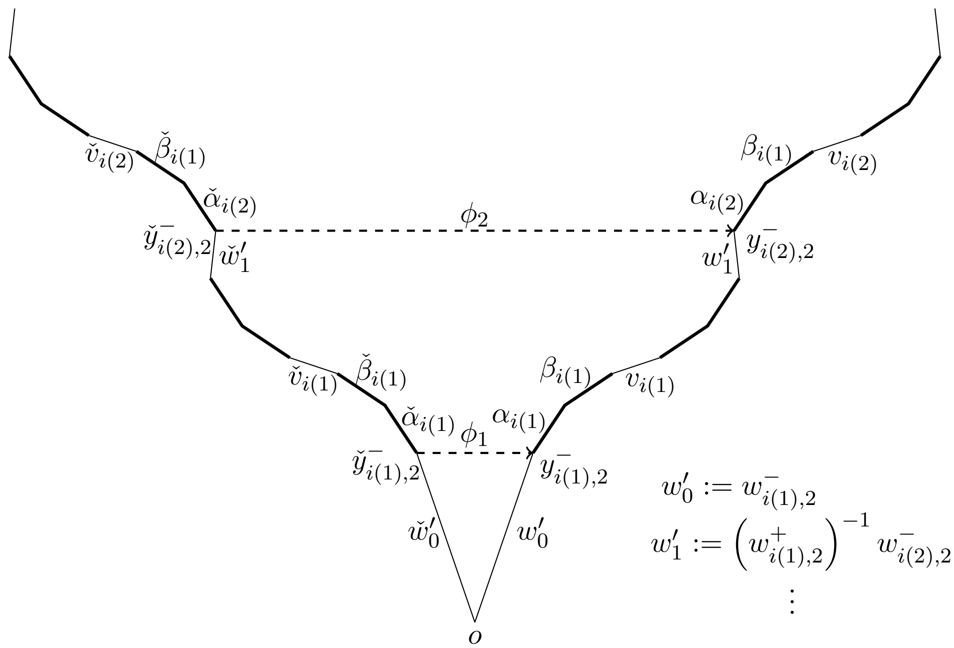

FIGURE 7. Defining ϕ_k's used in the pivoting for a pair of independent paths.

We now define

$$\begin{aligned} S_k^*(s, \check{s}) &:= \{\alpha_{i(k)} \in S : (\phi_k^{-1} o, \Gamma(\alpha_{i(k)})) \text{ is } K_0\text{-aligned}\} \\ &:= \left\{\alpha_{i(k)} \in S : \left(\check{y}^-_{\check{i}(k),2}, \Upsilon(\alpha_{i(k)})\right) \text{ is } K_0\text{-aligned}\right\}, \\ \check{S}_k^*(s, \check{s}) &:= \left\{\check{\alpha}_{\check{i}(k)} \in S : \left(\phi_k a_{i(k)} o, \Gamma(\check{\alpha}_{\check{i}(k)})\right) \text{ is } K_0\text{-aligned}\right\} \\ &:= \left\{\check{\alpha}_{\check{i}(k)} \in S : \left(y^-_{i(k),1}, \Upsilon(\check{\alpha}_{\check{i}(k)})\right) \text{ is } K_0\text{-aligned}\right\}. \end{aligned}$$

Then the property of Schottky sets imply that $S \setminus S_k^*$, $S \setminus \check{S}_k^*$'s consist of at most 1 element each. Moreover, Lemma 3.3 says that $\left(\bar{\Upsilon}(\check{\alpha}_{\check{i}(k)}), \Upsilon(\alpha_{i(k)})\right)$ is D_0-aligned when $\alpha_{i(k)} \in S_k^*$ and $\check{\alpha}_{\check{i}(k)} \in \check{S}_k^*$.

We now estimate the probability that $\alpha_{i(k)} \in S_k^*$ and $\check{\alpha}_{\check{i}(k)} \in \check{S}_k^*$. Given $s = (\alpha_{i(l)}, \beta_{i(l)}, \gamma_{i(l)})_{l=1,\ldots,k-1}$ and $\check{s} = (\check{\alpha}_{\check{i}(l)}, \check{\beta}_{\check{i}(l)}, \check{\gamma}_{\check{i}(l)})_{l=1,\ldots,k-1}$, we define

$$S_k^\dagger := \left\{ \begin{array}{c} (\alpha_{i(k)}, \beta_{i(k)}, \gamma_{i(k)}, \check{\alpha}_{\check{i}(k)}, \check{\beta}_{\check{i}(k)}, \check{\gamma}_{\check{i}(k)}) \in S_{i(k)}(\mathcal{E}) \times \check{S}_{\check{i}(k)}(\check{\mathcal{E}}) \\ : \ \alpha_{i(k)} \in S^*_{i(k)}(s) \text{ and } \check{\alpha}_{\check{i}(k)} \in \check{S}_k^*(\check{s}, \check{\alpha}_{\check{i}(k)}) \end{array} \right\}$$

Then we have the following:

Lemma 4.6. *For each $1 \le k \le \lfloor M/2 \rfloor$, $S_k^\dagger$ has cardinality at least $(\#S)^6 - 8(\#S)^5$.*

Proof. There are at least $(\#S-1)$ choices of $\gamma_{i(k)}$ and $\check{\gamma}_{\check{i}(k)}$ that satisfy Inequality 4.2. Fixing those choices, at least $(\#S-1)$ choices of $\beta_{i(k)}$ and $\check{\beta}_{i(\check{k})}$ in S satisfy Inequality 4.4. Fixing those choices, there are at most 1 choice of $\alpha_{i(k)}$ in S that violates Inequality 4.5 and at most 1 choice that lies outside S_k^*. If we choose $\alpha_{i(k)}$ in S_k^* that satisfies Inequality 4.5, now at least $(\#S-2)$ choices of $\check{\alpha}_{\check{i}(k)}$ satisfy Inequality 4.5 and belong to $\check{S}_k^*$. Overall, we conclude that $S_{(k)}^*$ has cardinality at least $(\#S-1)^4(\#S-2)^2 \geq (\#S)^6 - 8(\#S)^5$. ∎

Corollary 4.7. *If $\#P_n(\mathcal{E})$, $\#P_n(\check{\mathcal{E}})$ are greater than m, then we have*

$$\mathbb{P}\left(\alpha_{i(k)} \in S_k^*(s,\check{s}), \check{\alpha}_{\check{i}(k)} \in \check{S}_k^*(s,\check{s}) \text{ for some } k \leq m \,\middle|\, \mathcal{E} \times \check{\mathcal{E}}\right) \geq 1 - \left(\frac{8}{N_0}\right)^m. \tag{4.7}$$

4.3. **Pivotal times in random walks.** Let μ_S be the uniform measure on S. By taking suitably small α between 0 and 1, we can decompose μ^{4M_0} as

$$\mu^{4M_0} = \alpha\mu_S^4 + (1-\alpha)\nu$$

for some probability measure ν. We then consider:

- Bernoulli RVs ρ_i with $\mathbb{P}(\rho_i = 1) = \alpha$ and $\mathbb{P}(\rho_i = 0) = 1-\alpha$,
- η_i with the law μ_S^4, and
- ν_i with the law ν,

all independent, and define

$$(g_{4M_0k+1}, \ldots, g_{4M_0k+4M_0}) = \begin{cases} \nu_k & \text{when } \rho_k = 0, \\ \eta_k & \text{when } \rho_k = 1. \end{cases}$$

Then $(g_i)_{i=1}^{\infty}$ has the law μ^{∞}. We now define Ω to be the ambient probability space on which the above RVs are all measurable. We will denote an element of Ω by ω. We also fix

- $\omega_k := g_1 \cdots g_k$,
- $\mathscr{B}(k) := \sum_{i=0}^{k} \rho_i$, i.e., the number of the Schottky slots till k, and
- $\vartheta(i) := \min\{j \geq 0 : \mathscr{B}(j) = i\}$, i.e., the i-th Schottky slot.

For each $\omega \in \Omega$ and $i \geq 1$ we define

$$\begin{aligned} w_{i-1} &:= g_{4M_0[\vartheta(i-1)+1]+1} \cdots g_{4M_0\,\vartheta(i)}, \\ \alpha_i &:= (g_{4M_0\,\vartheta(i)+1}, \ldots, g_{4M_0\,\vartheta(i)+M_0}), \\ \beta_i &:= (g_{4M_0\,\vartheta(i)+M_0+1}, \ldots, g_{4M_0\,\vartheta(i)+2M_0}), \\ \gamma_i &:= (g_{4M_0\,\vartheta(i)+2M_0+1}, \ldots, g_{4M_0\,\vartheta(i)+3M_0}), \\ \delta_i &:= (g_{4M_0\,\vartheta(i)+3M_0+1}, \ldots, g_{4M_0\,\vartheta(i)+4M_0}). \end{aligned}$$

In other words, $\eta_{\vartheta(i)}$ corresponds to $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ (with M_0 steps each) and w_i corresponds to the products of intermediate steps of ν_k's in between $\eta_{\vartheta(i-1)}$ and $\eta_{\vartheta(i)}$. As in Subsection 4.1, we employ the notation $a_i := \Pi(\alpha_i)$, $b_i := \Pi(\delta_i)$ and so on.

In order to represent ω_n for arbitrary n, we set $n' := \lfloor n/4M_0 \rfloor$ and $w^{(n)} := g_{4M_0[\vartheta(\mathscr{B}(n'))+1]+1} \cdots g_n$. We then have

$$\omega_n = w_0 a_1 b_1 c_1 d_1 w_1 \cdots a_{\mathscr{B}(n')} b_{\mathscr{B}(n')} c_{\mathscr{B}(n')} d_{\mathscr{B}(n')} w^{(n)} \tag{4.8}$$

and we can bring the discussion in Subsection 4.1 here (with v_i's set as id). As before, we denote by s the choices of $\alpha_i, \beta_i, \gamma_i, \delta_i$ and define

$$\begin{aligned} P_1(\omega) &= P_1\left((w_i)_{i=0}^1, a_1, b_1, c_1, d_1\right), \\ P_2(\omega) &= P_2\left((w_i)_{i=0}^2, (a_i, b_i, c_i, d_i)_{i=1}^2\right), \\ &\vdots \end{aligned} \tag{4.9}$$

and

$$P^{(n)}(\omega) = P_{\mathscr{B}(n')}\left(\left(w_0, \ldots, w_{\mathscr{B}(n')-1}, w^{(n)}\right), (a_i, b_i, c_i, d_i)_{i=1}^{\mathscr{B}(n')}\right).$$

Note that $P^{(n)}(s)$ is built using the decomposition in Equation 4.8, and its partial sets of pivotal times are $P_1(\omega)$, ..., $P_{\mathscr{B}(n')-1}(\omega)$. We finally define

$$\mathcal{P}_n(\omega) := \left\{4M_0\,\vartheta(i) : i \in P^{(n)}(s)\right\}.$$

Lemma 4.8. *Let ω be a non-elementary random walk on G. Then $\mathcal{P}_n(\omega)$ increases linearly outside a set of exponentially decaying probability. More precisely, there exists $K > 0$ such that*

$$\mathbb{P}\left(\#\,\mathcal{P}_m(\omega) - \#\,\mathcal{P}_m(\omega) \le K(m-n)\right) \le \frac{1}{K} e^{-K(m-n)}$$

holds for all $0 \le n \le m$.

Proof. We denote $\lfloor m/4M_0 \rfloor$ by m' and $\lfloor n/4M_0 \rfloor$ by n'. Recall that the first model involves independent RVs $\{\rho_i, \eta_i, \nu_i\}$'s. We first draw choices of $\{\rho_i\}_{i=1}^m$ that determine the values of $\mathscr{B}(n')$ and $\{\vartheta(1), \ldots, \vartheta(\mathscr{B}(n'))\}$. Since ρ_i has uniform exponential moment and uniform positive expectation, $\mathscr{B}(n')$ increases linearly outside a set of exponentially decaying probability. More precisely, there exists K_1 (independent of m, n) such that for any m, n,

$$\mathbb{P}\left(\mathscr{B}\left(m'\right) - \mathscr{B}\left(n'\right) \le K_1(m-n)\right) \le \frac{1}{K_1} e^{-K_1(m-n)}. \tag{4.10}$$

Let us fix choices of $\{\rho_i\}_{i=1}^m$ that makes $\mathscr{B}(m') - \mathscr{B}(n') > K_1(m-n)$.

We then draw choices of $\{\nu_i\}_{i=1}^m$ that determine the values of $\{w_{i-1}\}_{i=1}^{\mathscr{B}(m')}$, $w'_{\mathscr{B}(m')}$ and $w'_{\mathscr{B}(n')}$. Now the values of $\{\alpha_i, \beta_i, \gamma_i, \delta_i\}_{i=1}^{\mathscr{B}(m')}$ are determined by the values of $\{\eta_{\vartheta(1)}, \ldots, \eta_{\vartheta(\mathscr{B}(m'))}\}$, which follow the law of $\mu_S^{4\,\mathscr{B}(n')}$. Now Corollary 4.5 provides a constant $K_2 > 0$ such that the following holds:

$$\begin{aligned} &\mathbb{P}\left(\#\,\mathcal{P}_m(\omega) - \#\,\mathcal{P}_n(\omega) \le K_2(m-n)\right) \\ &\le \mathbb{P}\left(\#P^{(m)}(\omega) - \#P_{\mathscr{B}(n')-1}(\omega) \le K_2(m-n) + 1\right) \\ &\le \frac{1}{K_2} e^{-K_2(\mathscr{B}(m') - \mathscr{B}(n'))} \le \frac{1}{K_2} e^{-K_2 K_1(m-n)}. \end{aligned}$$

Here, the first inequality is due to the relationship

$$\#\mathcal{P}_n(\omega) = \#P^{(n)}(\omega) \le \#P_{\mathscr{B}(n')-1}(\omega) + 1.$$

Combined with Inequality 4.10, this yields the desired conclusion. □

We now arrive at the first description of the escape rate.

Corollary 4.9. *Let ω be a non-elementary random walk on G. Then there exists $K > 0$ such that*

$$\mathbb{P}\left(d(o, \omega_n\, o) \le Kn\right) \le \frac{1}{K} e^{-Kn}.$$

Proof. Lemma 4.1 tells us that there exists a sequence of Schottky axes $(\kappa_l)_{l=1}^{M}$ with $M > 4\#\mathcal{P}_n(\omega)$ such that $(o, \kappa_1, \ldots, \kappa_M, \omega_n\, o)$ is D_0-aligned. Proposition 3.6 then tells us that

$$d(o, \omega_n) \ge \left[\left(\frac{M_0}{K_0} - K_0\right) - 3E_0\right] \cdot (4\#\mathcal{P}_n(\omega)) \ge 4E_0 \#\mathcal{P}_n(\omega).$$

By combining this with Lemma 4.8, we deduce the desired conclusion. □

Corollary 4.9 even implies

$$\mathbb{P}\left(\min\{\#\mathcal{P}_k(\omega) : k \ge n\} \le Kn\right) \le \frac{1}{K} e^{-Kn}$$

for some $K > 0$. We now claim that if $\#\mathcal{P}_k(\omega) \ge Kn$ for all $k \ge n$, then $\mathcal{P}_n(\omega)$, $\mathcal{P}_{n+1}(\omega)$, ... all possess the same first $Kn - 1$ pivotal times. Suppose to the contrary that for some $k \ge n$, $\mathcal{P}_k(\omega)$ does not start with the first $Kn - 1$ pivotal times of $\mathcal{P}_n(\omega)$.

Since the complete set of pivotal times $P^{(n)}(\omega)$ has at least Kn elements, $P_{\mathscr{B}(n')-1}(\omega)$ has at least $Kn - 1$ elements. Let $i_1, \ldots, i_{\lceil Kn-1 \rceil}$ be the first $\lceil Kn - 1 \rceil$ elements of $P_{\mathscr{B}(n')-1}$.

We now note that one of $\{P_{\mathscr{B}(n')}(\omega), \ldots, P_{\mathscr{B}(k')-1}(\omega), P^{(k)}(\omega)\}$ becomes a proper subset of $\{i_1, \ldots, i_{\lceil Kn-1 \rceil}\}$; otherwise all of $i_1, \ldots, i_{\lceil Kn-1 \rceil}$ survives in $P^{(k)}(\omega)$ and leads to a contradiction. If $P^{(k)}$ is so, then we have a contradiction $\#\mathcal{P}_k(\omega) = \#P^{(k)}(\omega) < Kn - 1$. Now suppose that $P_l(\omega)$ is so for some $\mathscr{B}(n') \le l < k$. Since $P_l(\omega) = P^{(4M_0\, \vartheta(l))}(\omega)$, we have

$$\#\mathcal{P}_{4M_0\, \vartheta(l)}(\omega) = \#P_l < Kn - 1, \tag{4.11}$$

$$\#\mathcal{P}_{4M_0\, \vartheta(l+1)}(\omega) = \#P_{l+1} \le P_l(\omega) + 1 < Kn. \tag{4.12}$$

Note that $4M_0\, \vartheta(l+1) > n$, since otherwise we have a contradiction, namely, $\mathscr{B}(\lfloor n/4M_0 \rfloor) \ge l+1 > \mathscr{B}(n')$. However, Inequality 4.12 then also contradicts the assumption. Hence, the claim follows.

Having the argument above in mind, we define

$$\mathcal{Q}_n(\omega) := \cap_{k \ge n} \mathcal{P}_k(\omega), \quad \mathcal{Q}(\omega) := \cup_n \mathcal{Q}_n(\omega) = \liminf \mathcal{P}_n(\omega)$$

; we call this the *set of eventual pivotal times.* We have proven:

Lemma 4.10.

$$\mathbb{P}(\#\,\mathcal{Q}_n(\omega) \le Kn) \le \frac{1}{K} e^{-Kn} \tag{4.13}$$

holds for some $K > 0$.

Suppose now that $\#\,\mathcal{Q}_n(\omega) = \{i(1) < \ldots < i(M)\}$. Let $(\kappa_l)_{l=1}^{4M}$ be the sequence of Schottky axes at pivotal times in $\mathcal{Q}_n(\omega)$. Then for any $k, k' \ge n$,

$$(o, \kappa_1, \ldots, \kappa_{4M}, \omega_k\, o), (o, \kappa_1, \ldots, \kappa_{4M}, \omega_{k'}\, o)$$

are subsequences of D_0-aligned sequences; namely, they are D_1-aligned. Then the terminating point $\omega_{i(M)+4M_0}\, o$ of the last axes κ_{4M} is far from o and passed by $[o, \omega_k\, o]$ and $[o, \omega_{k'}\, o]$. More precisely, we have

$$d(o, \omega_{i(M)+4M_0}\, o), d(\omega_{i(M)+4M_0}\, o, o) \ge \left[\left(\frac{M_0}{K_0} - K_0\right) - 3E_0\right] \cdot 4M \ge 4E_0 M$$

and

$$d(\omega_{i(M)+4M_0}\, o, [\omega_k\, o, o]), d(\omega_{i(M)+4M_0}\, o, [o, \omega_{k'}\, o]) \le E_0.$$

This implies that the Gromov product $(\omega_k\, o, \omega_{k'}\, o)_o$ is at least $4E_0 M - 5E_0$, and we have:

Corollary 4.11 ([Gou21, Proposition 4.13])**.** *There exists $K > 0$ such that the following hold:*

$$\mathbb{P}\left(\inf_{k,k' \ge n} (\omega_k\, o, \omega_{k'}\, o)_o \le Kn\right) \le \frac{1}{K} e^{-n/K}. \tag{4.14}$$

By combining subadditive ergodic theorem with Corollary 4.9, we obtain Theorem D for random walks with finite first moment. Gouëzel's arguments in [Gou21, Section 5] provide the remaining information; namely, the arguments there show that:

(1) Equation 1.3 holds with $\lambda = +\infty$ if μ has infinite first moment, and
(2) Theorem E holds.

Using the pivotal time construction in Section 4, one can realize Gouëzel's arguments in the current setting. Since the idea is identical, we will not repeat them here.

5. Deviation inequalities

In this section, we establish deviation inequalities and their consequences. In order to derive deviation inequalities, we will seek an (eventual) pivotal time at which the Schottky segment will witness two sides of the triangle made by points. This will make the triangle 'thin' and guarantee that the Gromov product is bounded by the progress made till the pivotal time. Such a pivotal time will appear before the n-th step outside a set of exponentially decaying probability. Using this exponential bound, we will estimate the p-moment and the $2p$-moment of the Gromov product.

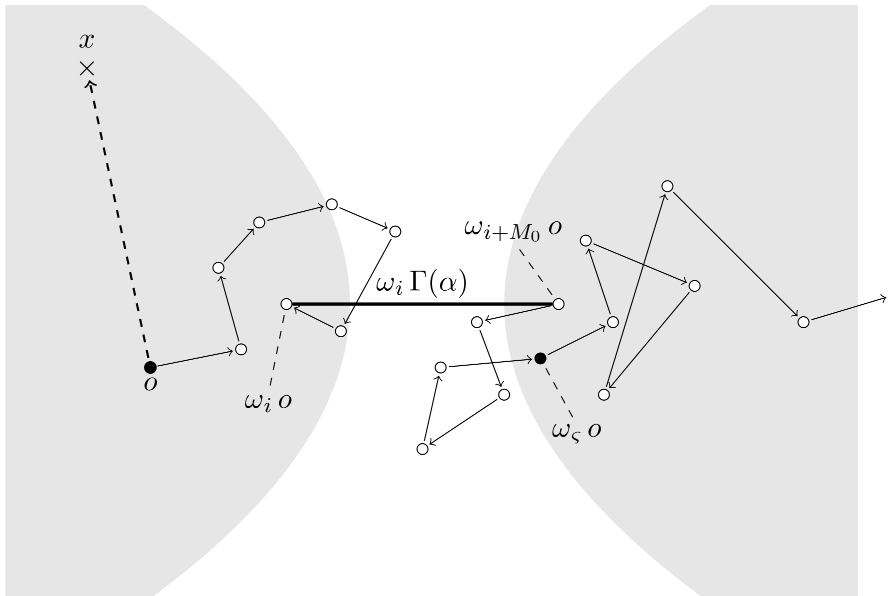

FIGURE 8. Persistent progress and ς. Here, o and x are on the left with respect to the persistent progress $\omega_i\,\Gamma(\alpha)$, while the loci after $\omega_\varsigma\, o$ are all on the right. Note that we do not restrict the locations of $\omega_1\, o, \ldots, \omega_{i-1}\, o$ and $\omega_{i+M_0+1}\, o, \ldots, \omega_{\varsigma-1}\, o$.

5.1. **Persistent progress.** Given $x \in X$, we seek an index k such that there exists $i \le k - M_0$ such that:

(1) $\alpha := (g_{i+1}, \ldots, g_{i+M_0})$ is a Schottky sequence;
(2) $(o, \omega_i\,\Gamma(\alpha), \omega_n\, o)$ is D_1-aligned for all $n \ge k$;
(3) $(x, \omega_i\,\Gamma(\alpha))$ is D_1-aligned.

Let $\varsigma = \varsigma(\omega; x)$ be the minimal index k that satisfies the above. If such an index does not exist, then we set $\varsigma = +\infty$.

For example, when $x = o$, $\varsigma(\omega; o)$ will be smaller than n if $\mathcal{Q}_n(\omega) \neq \emptyset$. We have previously constructed the pivotal times in order to guarantee witnessing of $[o, \omega_n\, o]$. We will now perform additional pivoting at the pivotal times in order to guarantee the witnessing of $[x, \omega_n\, o]$ as well.

Lemma 5.1. *There exists $K, \kappa > 0$ such that for any $x \in X$ and $g_{k+1} \in G$, we have*

$$\mathbb{P}\left(\varsigma(\omega; x) \ge k \,\middle|\, g_{k+1}\right) \le K e^{-\kappa k}$$

for each k.

Proof. The proof is essentially given in [Cho21a], except that we employ the language of BGIP and closest point projections here.

We first freeze the choices of $g_{4M_0\lfloor k/4M_0\rfloor+1}, \ldots, g_{4M_0(\lfloor k/4M_0\rfloor+1)}$ (or equivalently, the values of $\rho_{\lfloor k/4M_0\rfloor}$, $\nu_{\lfloor k/4M_0\rfloor}$ and $\eta_{\lfloor k/4M_0\rfloor}$) and exclude them from the potential pivotal time. We still have $\mathbb{P}(\#\,\mathcal{Q}_k \le \kappa_1 k) \le K_1 e^{-\kappa_1 k}$.

Let us fix an equivalence class $\mathcal{E}$ made by pivoting the choice of β_i's at the first $\kappa_1 k$ eventual pivotal times that appeared before k. Let $i(1) < \ldots < i(\kappa_1 k)$ be the first $\kappa_1 k$ eventual pivotal times in $\mathcal{Q}_k(\mathcal{E})$, and $j(1) < \ldots < j(\kappa_1 k)$ be be the corresponding indices in the fixed words model, i.e., $4M_0\,\vartheta(j(l)) = i(l)$ for $l = 1, \ldots, \kappa_1 k$.

Recall that $\omega \in \mathcal{E}$ is then determined by the choices $(\beta_{j(1)}, \ldots, \beta_{j(\kappa_1 k)})$, and at each l there are at least $N_0 - 1$ choices of $\beta_{j(l)}$ for the pivoting. For any $\omega \in \mathcal{E}$ and $l = 1, \ldots, \kappa_1 k$, we have:

- $i(l) + 3M_0 \le k - M_0$,
- $\beta_{j(l)} = (g_{i(l)+M_0+1}, \ldots, g_{i(l)+2M_0})$ and $\gamma_{j(l)} = (g_{i(l)+2M_0+1}, \ldots, g_{i(l)+3M_0})$ are Schottky, and
- $(o, \Upsilon(\beta_{j(l)}), \omega_n\, o)$, $(o, \Upsilon(\gamma_{j(l)}), \omega_n\, o)$ are D_1-aligned for all $n \ge k$ by Lemma 4.1 and Proposition 3.5.

It now suffices to guarantee for most $\omega \in \mathcal{E}$ that $(x, \Upsilon(\beta_{j(l)}))$ or $(x, \Upsilon(\gamma_{j(l)}))$ is D_1-aligned at some l.

Suppose that $(x, \Upsilon(\gamma_{j(\kappa_1 k)}))$ is not D_1-aligned for some $\omega \in \mathcal{E}$ $(*)$. Recall:

$$\big(\Upsilon(\alpha_{j(1)}), \Upsilon(\beta_{j(1)}), \Upsilon(\gamma_{j(1)}), \ldots, \Upsilon(\gamma_{j(\kappa_1 k)})\big)$$

is a subsequence of a D_0-aligned Schottky axes. $(*)$ and Proposition 3.5 imply that $(\Upsilon(\beta_{j(l)}), x)$ is D_1-aligned for $l = 1, \ldots, \kappa_1 k$. In particular, $(x, \Upsilon(\beta_{j(l)}))$ is not D_1-aligned for $l = 1, \ldots, \kappa_1 k$. Let us now consider

$$\tilde{\omega} = (\tilde{\beta}_{j(1)}, \ldots, \tilde{\beta}_{j(1)}) \in \mathcal{E}$$

that differs from ω. Let $j(l)$ be the first index at which ω and $\tilde{\omega}$ differ. Then $\omega_{i(l)+M_0} = \tilde{\omega}_{i(l)+M_0}$ holds, and $(x, \Upsilon(\tilde{\beta}_{j(l)}))$ is K_0-aligned by the property of the Schottky set S. Therefore, we have either:

- $(x, \Upsilon(\gamma_{j(\kappa_1 k)}))$ is D_1-aligned for all $\omega \in \mathcal{E}$, or;
- $(x, \Upsilon(\beta_{j(l)}))$ is K_0-aligned at some l for all but one $\omega \in \mathcal{E}$.

In summary, we have

$$\mathbb{P}\left(\varsigma(\omega; x) \ge k \,\middle|\, \mathcal{E}\right) \le \frac{1}{\#\mathcal{E}} \le \left(\frac{1}{N_0 - 1}\right)^{\kappa_1 k} \le \left(\frac{2}{N_0}\right)^{\kappa_1 k}.$$

These conditional probabilities and the probability $\mathbb{P}\{\#\,\mathcal{Q}_k(\omega) \le \kappa_1 k\}$ together take up an exponentially decaying probability. $\square$

For $\omega \in \Omega$ and $n, k \ge \varsigma(\omega; x)$, we have i such that

(1) $\alpha := (g_{i+1}, \ldots, g_{i+M_0})$ is a Schottky sequence;
(2) $(o, \omega_i\, \Gamma(\alpha), \omega_n\, o)$ and $(o, \omega_i\, \Gamma(\alpha), \omega_k\, o)$ are D_2-aligned, and
(3) $(x, \omega_i\, \Gamma(\alpha))$ is D_2-aligned.

By Proposition 3.6, there exists $q \in [x, \omega_n\, o]$ that are within d-distance E_0 from $\omega_i\, o$. Hence, we have

$$\begin{aligned}(x, \omega_n\, o)_o &\le \frac{1}{2}\left[\begin{array}{c} d(x, \omega_i\, o) + d(\omega_i\, o, o) + d(o, \omega_i\, o) + d(\omega_i\, o, \omega_n\, o) \\ -d(x, q) - d(q, \omega_n\, o) \end{array}\right] \\ &\le d(o, \omega_i\, o) + d(q, \omega_i\, o) < d(o, \omega_k\, o).\end{aligned}$$

Here, the final inequality holds because $[o, \omega_k\, o]$ is E_0-witnessed by $[\omega_i\, o, \omega_{i+M_0}\, o]$ whose length is at least $10E_0$.

For a similar reason, we have $d(o, [x, \omega_n\, o]) \le d(o, \omega_\varsigma\, o)$. Hence, we obtain:

Corollary 5.2. *There exist $\kappa, K > 0$ such that for any $x \in X$ and $g_{k+1} \in G$, we have*

$$\mathbb{P}\left[\sup_{n \ge k} (x, \omega_n\, o)_o \ge d(o, \omega_k\, o) \,\middle|\, g_{k+1}\right] \le K e^{-\kappa K},$$

$$\mathbb{P}\left[\sup_{n \ge k} d(o, [x, \omega_n\, o]) \ge d(o, \omega_k\, o) \,\middle|\, g_{k+1}\right] \le K e^{-\kappa K}.$$

Let us now define another index for a persistent progress made by two independent paths $(\check{\omega}, \omega)$. Given k, we seek an index $i \le k - M_0$ such that:

(1) $\alpha := (g_{i+1}, \ldots, g_{i+M_0})$ is a Schottky sequence;
(2) $(o, \omega_i\, \Gamma(\alpha), \omega_n\, o)$ is D_1-aligned for all $n \ge k$, and
(3) $(\check{\omega}_{n'}\, o, \omega_i\, \Gamma(\alpha))$ is D_2-aligned for all $n' \ge 0$.

We define $\upsilon = \upsilon(\check{\omega}, \omega)$ by the minimal index k such that the above index $i \le k$ exists. In other words, after index k, the forward path ω deviates forever from the directions made by each point in the backward path $\check{\omega}$. Moreover, this deviation is witnessed by some Schottky progress $\omega_i\, \Gamma(\alpha)$ made before index k.

Lemma 5.3. *There exist $\kappa, K > 0$ such that the following estimate holds for all k:*

$$\mathbb{P}\left(\upsilon(\check{\omega}, \omega) \ge k \,\middle|\, g_{k+1}, \check{g}_1, \ldots, \check{g}_{k+1}\right) \le K e^{-\kappa k}. \tag{5.1}$$

Proof. We first freeze the choices of $g_{4M_0\lfloor k/4M_0 \rfloor + 1}, \ldots, g_{4M_0(\lfloor k/4M_0 \rfloor + 1)}$ and $\check{g}_1, \ldots, \check{g}_{4M_0\lceil (k+1)/4M_0 \rceil}$. We still have $\mathbb{P}(\#\, \mathcal{Q}_k(\omega) \le \kappa_1 k) \le K_1 e^{-\kappa_1 k}$ and $\mathbb{P}(\#\, \mathcal{Q}_{2k}(\check{\omega}) \le \kappa_1 k) \le K_1 e^{-\kappa_1 k}$.

Now for paths ω with $\mathcal{Q}_k(\omega) > \kappa_1 k$, we pivot at the first $\kappa_1 k$ pivotal times; let $\mathcal{E}$ be one equivalence class made from this early pivoting. Let also $\check{\mathcal{E}}$ be an equivalence class of backward paths $\check{\omega}$'s that have $\#\, \mathcal{Q}_{2k}(\check{\omega}) \ge \kappa_1 k$, made by pivoting at the first $\kappa_1 k$ pivotal times. Note that the pivotal times for $\check{\omega}$'s are always formed after $k+1$ since we have frozen the first $4M_0\lceil (k+1)/4M_0 \rceil$ steps. Let

$$\begin{aligned} \mathcal{Q}_k(\mathcal{E}) &= \{i(1) < \ldots < i(\kappa_1 k) < \ldots\}, \\ \mathcal{Q}_{2k}(\check{\mathcal{E}}) &= \{\check{i}(1) < \ldots < \check{i}(\kappa_1 k) < \ldots\}, \\ i(l) = 4M_0\, \vartheta(j(l)), &\quad \check{i}(l) = 4M_0 \check{\vartheta}(\check{j}(l)) \quad (l = 1, \ldots, \kappa_1 k). \end{aligned}$$

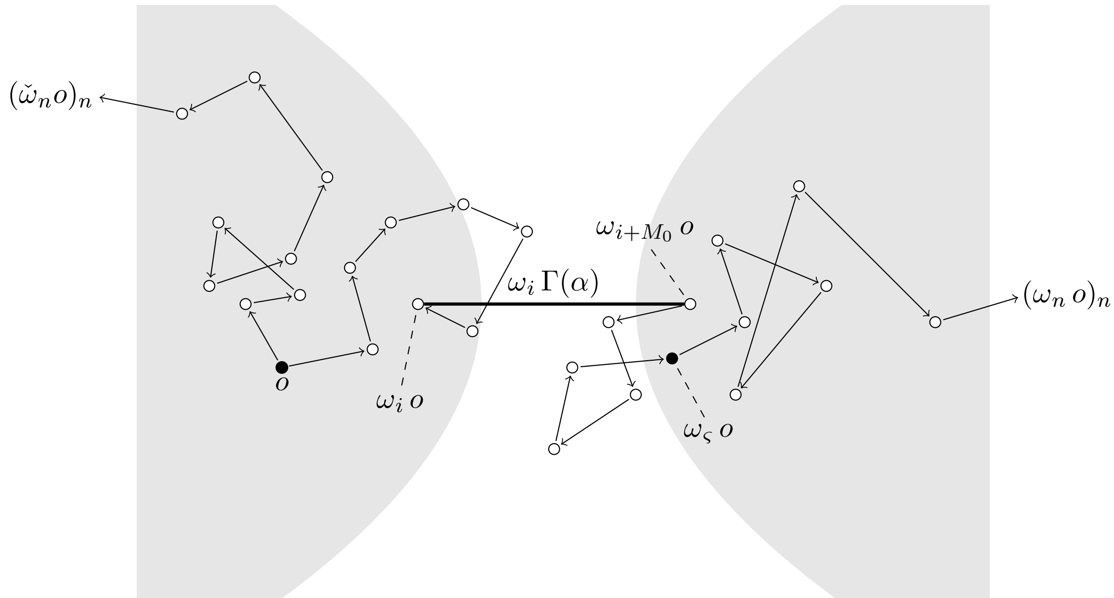

FIGURE 9. Persistent progress and υ. Here, all of the backward loci $(\check{\omega}_n o)_{n\geq 0}$ are on the left of the persistent progress $\omega_i\,\Gamma(\alpha)$, while the forward loci after $\omega_\varsigma\, o$ are all on the right.

Now on $\check{\mathcal{E}}\times\mathcal{E}$, Corollary 4.7 implies that $(\bar{\Upsilon}(\check{\alpha}_{\check{j}(l)}), \Upsilon(\alpha_{j(l)}))$ is K_0-aligned for some $l \leq \kappa_1 k/2$ for probability at least $1-(8/N_0)^{\kappa_1 k/2}$ on $\check{\mathcal{E}}\times\mathcal{E}$. We now freeze the choices at the first $\kappa_1 k/2$ pivotal times for ω and the entire pivotal times for $\check{\omega}$ that make $(\bar{\Upsilon}(\check{\alpha}_{\check{j}(l)}), \Upsilon(\alpha_{j(l)}))$ K_0-aligned. Then $\mathcal{E}$ is divided into finer equivalence classes $\mathcal{E}_1$ made by pivoting at the latter $\kappa_1 k/2$ pivotal times.

Lemma 5.1 asserts that for each $n' = 1, 2, \ldots, 2k$, $(\check{\omega}_{n'}\, o, \Upsilon(\gamma_{j(\kappa_1 k)}))$ is D_1-aligned for all but at most one choice in $\mathcal{E}_1$. Except at most $2k$ such bad choices, we now have the following:

- $i(\kappa_1 k) + 4M_0 \leq k$,
- $(o, \Upsilon(\gamma_{j(\kappa_1 k)}), \omega_n\, o)$ is D_2-aligned for all $n \geq k$,
- $\big(\check{\omega}_{n'}\, o, \bar{\Upsilon}(\check{\gamma}_{\check{j}(l)}), \Upsilon(\alpha_{j(l)}), \Upsilon(\gamma_{j(\kappa_1 k)})\big)$ is a subsequence of a D_1-aligned sequence for all $n' \geq 2k$, and
- $\big(\check{\omega}_{n'}\, o, \Upsilon(\gamma_{j(\kappa_1 k)})\big)$ is D_2-aligned for $n' = 1, \ldots, 2k$.

Then $(\check{\omega}_{n'} o, \Upsilon(\gamma_{j(\kappa_1 k)}))$ is D_2-aligned for all n' by Proposition 3.5, and $i(\kappa_1 k) + 2M_0 \leq k - M_0$ works for ω. Hence,

$$\mathbb{P}\left(\upsilon(\check{\omega}, \omega) \geq k \,\middle|\, \check{\mathcal{E}}\times\mathcal{E}\right) \leq \left(\frac{8}{N_0}\right)^{\kappa_1 k/2} + 2k\cdot\left(\frac{3}{N_0}\right)^{\kappa_1 k/2}.$$

We now sum up these conditional probabilities and the excluded probability to conclude. $\square$

As before, we deduce

$$(\check{\omega}_{n'}o, \omega_n\, o)_o \le d(o, \omega_k\, o)$$

for all $n' \ge 0$ and $n, k \ge \upsilon(\check{\omega}, \omega)$. Hence, we deduce:

Corollary 5.4. *There exist $\kappa, K > 0$ such that for any $g_{k+1}, \check{g}_1, \ldots, \check{g}_{k+1} \in G$, we have*

$$\mathbb{P}\left[\sup_{n' \ge 0, n \ge k} (\check{\omega}_{n'}o, \omega_n\, o)_o \ge d(o, \omega_k\, o) \,\middle|\, g_{k+1}, \check{g}_1, \ldots, \check{g}_{k+1}\right] \le Ke^{-\kappa k}.$$

We similarly define $\check{\upsilon} = \check{\upsilon}(\check{\omega}, \omega)$ as the minimal index k that are associated with another index $i \le k$ such that:

(1) $\check{\alpha} := (\check{g}_{i+1}, \ldots, \check{g}_{i+M_0})$ is a Schottky sequence;
(2) $(o, \check{\omega}_i\Gamma(\check{\alpha}), \check{\omega}_n o)$ is D_1-aligned for all $n \ge k$, and
(3) $(\omega_n\, o, \check{\omega}_i\Gamma(\check{\alpha}))$ is D_2-aligned for all $n \ge 0$.

Then we similarly have

$$\mathbb{P}\left(\check{\upsilon}(\check{\omega}, \omega) \ge k \,\middle|\, \check{g}_{k+1}, g_1, \ldots, g_{k+1}\right) \le K_2 e^{-\kappa_2 k}. \tag{5.2}$$

Note that Inequality 5.1 is proven using the pivoting at the first k steps of ω and eventual escape to infinity of ω, $\check{\omega}$. This enables us to fix $\check{g}_1, \ldots, \check{g}_{k+1}$ and g_{k+1} in prior: we do not use the randomness of the initial trajectory of $\check{\omega}$. Likewise, Inequality 5.2 does not rely on the pivoting at the initial k steps of ω. This will lead to the exponent doubling for the geodesic tracking; roughly speaking, this is a consequence of the fact that the minimum of two independent RVs with finite p-th moment has finite $2p$-th moment.

5.2. **Deviation inequalities.** Thanks to Corollary 5.2 and 5.4, we can establish the following deviation inequality.

Proposition 5.5. *Suppose that μ has finite p-moment for some $p > 0$. Then there exists $K > 0$ such that for any $x \in X$, we have*

$$\mathbb{E}\left[\sup_{n \ge 0} (x, \omega_n\, o)_o^p\right], \quad \mathbb{E}\left[\sup_{n, n' \ge 0} (\check{\omega}_{n'}o, \omega_n\, o)_o^{2p}\right] < K.$$

Note the difference between this proposition and [Cho21a, Proposition 5.6, 5.8]; we are taking the global suprema, not the limit suprema.

Proof. We have observed that $\sup_{n \ge \varsigma(\omega;x)} (x, \omega_n\, o) o$ is dominated by $d(o, \omega_{\varsigma(\omega;x)}\, o)$. Moreover, for $i = 1, \ldots, \varsigma(\omega; x)$, $(x, \omega_i\, o)_o$ and $(\omega_i\, o, x)_o$ are bounded above by $d(o, \omega_i\, o)$. Hence, we have

$$\begin{aligned} \sup_n (x, \omega_n\, o)_o^p &\le \max_{1 \le i \le \varsigma(\omega;x)} d(o, \omega_i\, o)^p \\ &\le \sum_{i=0}^{\infty} \left| d(o, \omega_{i+1}\, o)^p - d(o, \omega_i\, o)^p \right| 1_{i < \varsigma(\omega;x)}. \end{aligned}$$

Let us now recall that two simple inequalities: for $t, s \geq 0$,

$$|t^p - s^p| \leq \begin{cases} |t-s|^p & p \leq 1, \\ 2^p \left(|t-s|^p + s^{p-1}|t-s|\right) & p > 1. \end{cases} \tag{5.3}$$

Moreover, for $t_1, \ldots, t_n \geq 0$ and $p > 0$, we have

$$(t_1 + \ldots + t_n)^p \leq (n \max_i t_i)^p \leq n^p(t_1^p + \ldots + t_n^p)$$

and

$$\mathbb{E}[d(o, \omega_n\, o)^p] \leq n^{p+1}\, \mathbb{E}_\mu[d(o, go)^p].$$

Hence, it suffices to show that

$$\mathbb{E}\left[\sum_{i=0}^{\infty} d(o, g_{i+1}o)^p 1_{i<\varsigma(\omega;x)}\right] < K_1$$

for some K_1 that does not depend on x, and when $p > 1$, also

$$\mathbb{E}\left[\sum_{i=0}^{\infty} d(o, \omega_i\, o)^{p-1} d(o, g_{i+1}o) 1_{i<\varsigma(\omega;x)}\right] < K_2$$

for some K_2 that does not depend on x.

The first summation is estimated based on Lemma 5.1. Let K_3, κ_3 be as in Lemma 5.1; recall that K_3, κ_3 does not depend on x. Then we have

$$\begin{aligned} \sum_{i=0}^{\infty} \mathbb{E}\left[d(o, g_{i+1}o)^p 1_{i<\varsigma}\right] &= \sum_{i=0}^{\infty} \mathbb{E}\left[d(o, g_{i+1}o)^p \cdot \mathbb{P}\left(\varsigma(\omega; x) > i \,\middle|\, g_{i+1}\right)\right] \\ &\leq \sum_{i=0}^{\infty} \mathbb{E}\left[d(o, g_{i+1}o)^p \cdot K_3 e^{-\kappa_3 i}\right] \\ &\leq 2^p \left(\mathbb{E}_\mu[d(o, go)^p] + \mathbb{E}_{\check{\mu}}[d(o, go)^p]\right) \cdot K_3 \sum_i e^{-\kappa_3 i} =: K_1 \end{aligned}$$

Similarly, for $p > 1$, we estimate based on a dichotomy. Note that for any g_{i+1} and $c > 0$, we have

$$\begin{aligned} \mathbb{E}\left[d(o, \omega_i\, o)^{p-1} 1_{\varsigma > i} \,\middle|\, g_{i+1}\right] &\leq \mathbb{E}\left[d(o, \omega_i\, o)^{p-1} 1_{\varsigma > i} 1_{d(o, \omega_i\, o) \leq c} \,\middle|\, g_{i+1}\right] \\ &\quad + \mathbb{E}\left[d(o, \omega_i\, o)^{p-1} 1_{\varsigma > i} 1_{d(o, \omega_i\, o) > c} \,\middle|\, g_{i+1}\right] \\ &\leq c^{p-1}\, \mathbb{P}\left(\varsigma > i \,\middle|\, g_{i+1}\right) + \mathbb{E}\left[d(o, \omega_i\, o)^p \cdot c^{-1} | g_{i+1}\right] \\ &\leq c^{p-1} K_3 e^{-\kappa_3 i} + c^{-1} i^{p+1} \cdot \mathbb{E}_\mu[d(o, go)^p]. \end{aligned} \tag{5.4}$$

By setting $c = e^{\kappa_3 i/2p}$, we deduce

$$\begin{aligned}
&\sum_{i=0}^{\infty} \mathbb{E}\left[d(o,\omega_i\, o)^{p-1} d(o, g_{i+1}o) 1_{i<\varsigma}\right] \\
&= \sum_{i=0}^{\infty} \mathbb{E}\left[d(o,g_{i+1}o)\, \mathbb{E}\left[d(o,\omega_i\, o)^p 1_{\varsigma>i} \,\middle|\, g_{i+1}\right]\right] \\
&\le \sum_{i=0}^{\infty} \mathbb{E}\left[d(o,g_{i+1}o)\cdot \left(K_3 e^{-\kappa_3 i/2} + i^{p+1} e^{-\kappa_3 i/2p}\, \mathbb{E}_\mu[d(o,go)^p]\right)\right] \\
&\le \left(K_3\, \mathbb{E}_\mu[d(o,go)^p] + \mathbb{E}_\mu[d(o,go)^p]^2\right)\cdot \sum_i i^{p+1} e^{-\kappa i/2p} =: K_2.
\end{aligned}$$

Clearly, K_1 and K_2 do not depend on the choice of x.

We now investigate $(\check{\omega}_{n'}o, \omega_n\, o)_o$. Let

$$\check{D}_k := \sum_{i=1}^{k} d(o, \check{g}_i o), \quad D_k := \sum_{i=1}^{k} d(o, g_i o).$$

It is clear that $d(o,\omega_k\, o) < D_l$ for all $k \le l$.

We begin by claiming that

$$\sup_{n',n\ge 0} (\check{\omega}_{n'}o, \omega_n\, o)_o^{2p} \le \sum_{i=0}^{\infty} |\check{D}_{i+1}^p D_{i+1}^p - \check{D}_i^p D_i^p| \left(1_{\check{D}_i \ge D_i} 1_{i<\upsilon} + 1_{\check{D}_i \le D_i} 1_{i<\check{\upsilon}}\right).$$

First, note that the RHS is at least $\check{D}_l^p D_l^p$ for

$$l := \min\left\{i : 1_{\check{D}_i \ge D_i} 1_{i<\upsilon} + 1_{\check{D}_i \le D_i} 1_{i<\check{\upsilon}} = 0\right\}.$$

(If such minimum does not exist, then the RHS becomes infinity almost surely since $\check{D}_k$, D_k tends to infinity almost surely.) Note that either $\check{D}_l \ge D_l$ or $\check{D}_l \le D_l$ holds.

In the first case $l \ge \upsilon$ must hold. Then for $n' \ge 0$ and $n \ge l$, we have

$$(\check{\omega}_{n'}o, \omega_n\, o)_o^{2p} \le d(o,\omega_l\, o)^{2p} \le D_l^{2p} \le \check{D}_l^p D_l^p.$$

Moreover, for $n' \ge 0$ and $n \le l$, we have

$$(\check{\omega}_{n'}o, \omega_n\, o)_o^{2p} \le d(o,\omega_n\, o)^{2p} \le D_n^{2p} \le D_l^{2p} \le \check{D}_l^p D_l^p.$$

In the second case $l \ge \check{\upsilon}$ must hold, and the argument as above implies that $(\check{\omega}_{n'}o, \omega_n\, o)_o^{2p}$ is dominated by $\check{D}_l^p D_l^p$, as desired.

Note that for $t_i, s_i \ge 0$, we have

$$\begin{aligned}
|t_1^p t_2^p - s_1^p s_2^p| &= |t_1^p (t_2^p - s_2^p) + (t_1^p - s_1^p) s_2^p| \\
&\le 2^{p+q}\left(|t_1 - s_1|^p + s_1^{p-n_p}|t_1 - s_1|^{n_p} + s_1^p\right)\left(|t_2 - s_2|^p + s_2^{p-n_p}|t_2 - s_2|^{n_p}\right) \\
&\quad + 2^p\left(|t_1 - s_1|^p + s_1^{p-n_p}|t_1 - s_1|^{n_p}\right) s_2^p. \\
&\left(n_p = \begin{cases} p & 0 \le p \le 1 \\ 1 & p > 1. \end{cases}\right)
\end{aligned}$$

Considering this, it suffices to show

$$\mathbb{E}\left[d(o,\check{g}_{i+1})^{n_1} d(o,g_{i+1})^{n_2} \check{D}_i^{p-n_1} D_i^{p-n_2}\left(1_{\check{D}_i \geq D_i} 1_{i<\upsilon} + 1_{\check{D}_i \leq D_i} 1_{i<\check{\upsilon}}\right)\right] < K(i+1)^{2p+2} e^{-\kappa i}$$

for some K and κ, for $0 \leq n_1, n_2 \leq p$ such that $n_1 + n_2 \geq \min(p,1)$. We will discuss the case $n_2 > 0$; the other case can be handled in the same way.

Let us first fix $\check{g}_{i+1}$ and g_{i+1}. We then compute

$$\begin{aligned}
&\mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{\check{D}_i \geq D_i} 1_{i<\upsilon} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] \\
&\leq \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{D_i > c} 1_{\check{D}_i \geq D_i} 1_{i<\upsilon} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] + \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{D_i \leq c} 1_{\check{D}_i \geq D_i} 1_{i<\upsilon} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] \\
&\leq \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p} \cdot c^{-n_2} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] + \mathbb{E}\left[\check{D}_i^{p-n_1} \cdot \mathbb{E}\left[c^{p-n_2} 1_{i<\upsilon} \,\middle|\, \check{g}_1, \ldots, \check{g}_{i+1}, g_{i+1}\right]\right] \\
&\leq \mathbb{E}[\check{D}_i^{p-n_1}] \cdot \mathbb{E}[D_i^p] \cdot c^{-n_2} + \mathbb{E}[\check{D}_i^{p-n_1}] \cdot c^{p-n_2}\, \mathbb{P}\left[\upsilon > i \,\middle|\, \check{g}_1, \ldots, \check{g}_{i+1}, g_{i+1}\right] \\
&\leq (i+1)^{p-n_1+1}\, \mathbb{E}_\mu[d(o,go)^{p-n_1}] \cdot (i+1)^{p+1}\, \mathbb{E}_\mu[d(o,go)^p] \cdot c^{-n_2} \\
&\quad + (i+1)^{p-n_1+1}\, \mathbb{E}_\mu[d(o,go)^{p-n_1}] \cdot c^{p-n_2} \cdot K_3 e^{-\kappa_3 i}.
\end{aligned}$$

We also observe

$$\begin{aligned}
&\mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{\check{D}_i \leq D_i} 1_{i<\check{\upsilon}} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] \\
&\leq \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{\check{D}_i > c} 1_{\check{D}_i \leq D_i} 1_{i<\check{\upsilon}} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] + \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{\check{D}_i \leq c} 1_{\check{D}_i \leq D_i} 1_{i<\check{\upsilon}} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] \\
&\leq \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{D_i > c} 1_{i<\check{\upsilon}} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] + \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{\check{D}_i \leq c} 1_{i<\check{\upsilon}} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] \\
&\leq \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p} \cdot c^{-n_2} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] + \mathbb{E}\left[D_i^{p-n_2} \cdot \mathbb{E}\left[c^{p-n_1} 1_{i<\check{\upsilon}} \,\middle|\, \check{g}_{i+1}, g_1, \ldots, g_{i+1}\right]\right] \\
&\leq \mathbb{E}[\check{D}_i^{p-n_1}] \cdot \mathbb{E}[D_i^p] \cdot c^{-n_2} + \mathbb{E}[D_i^{p-n_2}] \cdot c^{p-n_1}\, \mathbb{P}\left[\check{\upsilon} > i \,\middle|\, \check{g}_{i+1}, g_1 \ldots, g_{i+1}\right] \\
&\leq (i+1)^{p-n_1+1}\, \mathbb{E}_\mu[d(o,go)^{p-n_1}] \cdot (i+1)^{p+1}\, \mathbb{E}_\mu[d(o,go)^p] \cdot c^{-n_2} \\
&\quad + (i+1)^{p-n_2+1}\, \mathbb{E}_\mu[d(o,go)^{p-n_2}] \cdot c^{p-n_1} \cdot K_3 e^{-\kappa_3 i}.
\end{aligned}$$

Note that the trick

$$\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{D_i > c} < \check{D}_i^{p-n_1} D_i^{p} c^{-n_2}$$

makes use of the fact $n_2 > 0$; it cannot work on the side of $\check{D}_i$ since n_1 may vanish in this case. Throughout the first argument, the factor $1_{\check{D}_i \geq D_i}$ did not play any role (though it is necessary for the case $n_1 > 0$ and $n_2 = 0$); the factor $1_{\check{D}_i \leq D_i}$ in the second argument played a role only once, namely, switching $\check{D}_i$ and D_i at the second step.

The proof ends by taking $c = e^{\kappa_3 i/2p}$. □

We now record a corollary for the geodesic tracking.

Corollary 5.6. *Suppose that μ has finite p-moment for some $p > 0$. Then there exists $K > 0$ such that*

$$\mathbb{E}\left[\min\{d(o,\omega_\upsilon\, o), d(o,\check{\omega}_{\check{\upsilon}} o)\}^{2p}\right] < K.$$

Proof. In view of the second half of the previous proof, it suffices to check

$$\min\{d(o,\omega_\upsilon\, o), d(o,\check{\omega}_{\check{\upsilon}}o)\}^{2p} \le \sum_{i=0}^{\infty} |\check{D}_{i+1}^p D_{i+1}^p - \check{D}_i^p D_i^p| \left(1_{\check{D}_i \ge D_i} 1_{i<\upsilon} + 1_{\check{D}_i \le D_i} 1_{i<\check{\upsilon}}\right).$$

The RHS is at least $\check{D}_l^p D_l^p$ for $l = \min\{i : 1_{\check{D}_i \ge D_i} 1_{i<\upsilon} + 1_{\check{D}_i \le D_i} 1_{i<\check{\upsilon}} = 0\}$. Note that either $\check{D}_l \ge D_l$ or $\check{D}_l \le D_l$ holds. In the first case, we are forced to have $l \ge \upsilon$; then

$$\min\{d(o,\omega_\upsilon\, o), d(o,\check{\omega}_{\check{\upsilon}}o)\}^{2p} \le d(o,\omega_\upsilon\, o)^{2p} \le D_\upsilon^{2p} \le D_l^{2p} \le \check{D}_l^p D_l^p.$$

In the second case, we are forced to have $l \ge \check{\upsilon}$; then

$$\min\{d(o,\omega_\upsilon\, o), d(o,\check{\omega}_{\check{\upsilon}}o)\}^{2p} \le d(o,\check{\omega}_{\check{\upsilon}}o)^{2p} \le \check{D}_{\check{\upsilon}}^{2p} \le \check{D}_l^{2p} \le \check{D}_l^p D_l^p$$

as desired. □

We also discuss the case of finite exponential moment.

Proposition 5.7. *Suppose that μ has finite exponential moment. Then there exist $\kappa, K > 0$ such that*

$$\mathbb{E}\left[\sup_{n,n'\ge 0} e^{\kappa(x,\omega_n\, o)_o}\right] < K, \quad \mathbb{E}\left[\sup_{n,n'\ge 0} e^{\kappa(\check{\omega}_{n'}o,\omega_n\, o)_o}\right] < K.$$

Proof. We explain the latter inequality; the former one follows from the same argument by replacing the role of υ with ς.

Note that $(\check{\omega}_n' o, \omega_n\, o)_o \le d(o,\omega_\upsilon\, o)$ for $n' \ge 0$ and $n \ge \upsilon(\check{\omega},\omega)$, and $(\check{\omega}_{n'}o,\omega_n\, o)_o \le d(o,\omega_n\, o) \le D_\upsilon$ for $0 \le n \le \upsilon(\check{\omega},\omega)$. This implies

$$\sup_{n,n'\ge 0} e^{\kappa(\check{\omega}_{n'}o,\omega_n\, o)_o} \le e^{\kappa D_\upsilon} \le \sum_{i=0}^{\infty} e^{\kappa D_i} 1_{i<\upsilon}.$$

Let us estimate the expectation of the summand. Fixing $\omega = (\check{g}_1, \check{g}_2, \ldots)$ and g_{i+1}, we observe

$$\begin{aligned} \mathbb{E}\left[e^{\kappa D_i} 1_{i<\upsilon}\right] &= \mathbb{E}\left[e^{\kappa D_i} 1_{D_i<c} 1_{i<\upsilon}\right] + \mathbb{E}\left[e^{\kappa D_i} 1_{D_i\ge c} 1_{i<\upsilon}\right] \\ &\le \mathbb{E}\left[e^{\kappa c} 1_{i<\upsilon}\right] + \mathbb{E}\left[e^{(A+1)\kappa D_i} e^{-A\kappa c}\right] \\ &\le e^{\kappa c} \cdot K_3 e^{-\kappa_3 i} + e^{-A\kappa c}\, \mathbb{E}_\mu \left[e^{(A+1)\kappa d(o,go)}\right]^i. \end{aligned} \tag{5.5}$$

By the assumption, $\mathbb{E}[e^{md(o,go)}] < M$ for some $m, M > 0$. We first take $c = c_1 i$ for each i, where c_1 is large enough so that $e^{c_1 m} \ge M^4$. We then take κ small enough so that $11\kappa < m$ and $\kappa c_1 < \kappa_3/4$, and $(A+1)\kappa = m$. Then the RHS of Inequality 5.5 decays exponentially as desired. □

5.3. **Central limit theorems.** We now prove Theorem C.

Proof. Let us first assume that μ has finite second moment; then Proposition 5.5 gives the uniform fourth-moment deviation inequality. Then Theorem 4.2 of [MS20] asserts that $[d(o, \omega_n\, o) - \lambda n]/\sqrt{n}$ converges to a Gaussian law in distribution.

If μ is non-arithmetic, there exists $g, g' \in \operatorname{supp} \mu^{*M}$ that has distinct translation lengths. By taking powers if necessary, we may assume that $d(o, go) - d(o, g'o) \geq 104E_0$. Now using the decomposition

$$\mu^{4M_0+M} = \alpha(\mu_S^2 \times (1/2_{\{g\}} + 1/2_{\{g'\}}) \times \mu_S^2) + (1-\alpha)\nu$$

and the argument for [Cho21a, Claim 6.1], we conclude that $Var[d(o, \omega_n)]$ increases at least linearly. This implies that the limiting distribution has strictly positive variance.

Now consider the inequality

$$|d(o, go) - d(o, ho)| \leq 104E_0.$$

If this holds for all $g, h \in \operatorname{supp} \mu^{*n}$ for all n, then we have

$$|d(o, go) - \lambda n| \leq 104E_0$$

for all $g \in \operatorname{supp} \mu^{*n}$ for all n and the limiting distribution will be degenerate. In other words, if the limiting distribution is non-degenerate, then there exist n and $g, h \in \operatorname{supp} \mu^{*n}$ such that $|d(o, go) - d(o, ho)| > 104E_0$. Since there are many choices in the Schottky set S, there exists $s \in S$ such that $(\Gamma(s), go)$ and $(\Gamma(s), ho)$ are K_0-aligned. Moreover, there exists $s' \in S$ such that $(g^{-1}(\Pi(s))^{-1}o, \Gamma(s'))$ and $(h^{-1}(\Pi(s))^{-1}o, \Gamma(s'))$ are K_0-aligned. These conditions imply that

$$(\Gamma(s), \Pi(s)g\Gamma(s')), (\Gamma(s), \Pi(s)h\Gamma(s'))$$

are D_0-aligned by Lemma 3.3, and consequently, that

$$(o, g\Gamma(s'), g\Pi(s')\Gamma(s), g\Pi(s')\Pi(s)g\Gamma(s'), \ldots, (g\Pi(s')g)^{n-1}g\Pi(s')\Gamma(s), (g\Pi(s')\Pi(s))^n o),$$
$$(o, g\Gamma(s'), h\Pi(s')\Gamma(s), h\Pi(s')\Pi(s)h\Gamma(s'), \ldots, (h\Pi(s')h)^{n-1}h\Pi(s')\Gamma(s), (h\Pi(s')\Pi(s))^n o)$$

are D_0-aligned. In particular, the Gromov products among the endpoints are bounded by E_0 and we deduce

$$\big|\tau(g\Pi(s)\Pi(s')) - d(o, go) + d(o, \Pi(s)o) + d(o, \Pi(s')o)]\big| \leq 3E_0,$$
$$\big|\tau(h\Pi(s)\Pi(s')) - d(o, ho) + d(o, \Pi(s)o) + d(o, \Pi(s')o)]\big| \leq 3E_0.$$

In summary, we obtained two elements $g\Pi(s)\Pi(s')$, $h\Pi(s)\Pi(s')$ in the support of $\mu^{*(n+2M_0)}$ whose translation lengths are distinct; μ is non-arithmetic.

For the LIL, we refer to the proof in [Cho21a, Section 7]. The proof there relies on the bounds on $\mathbb{E}[(\check{\omega}_m o, \omega_{m'}\, o)_o^3]$'s for various m, m', which we have for μ with finite second moment. $\square$

Let us observe a quantitative version of CLT.

Theorem 5.8. *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary, non-arithmetic measure μ on G. Suppose that μ has finite third moment, and let $F_n(x)$ be the distribution of $[d(o, \omega_n o) - n\lambda]/\sigma\sqrt{n}$. Then there exists $K > 0$ such that*

$$|F_n(x) - \mathcal{N}(x)| \le \frac{K}{\sqrt[5]{n}}$$

holds for all x and n.

Proof. Let us denote $\frac{1}{\sqrt{n}}\sqrt{Var[d(o, \omega_n o)]}$ by σ_n. In [Cho21a, Section 6], we proved the existence of a constant $K > 0$ such that the following hold. All descriptions are with respect to the Lévy metric.

(1) the RVs $\left(\frac{1}{\sqrt{n}} d(o, \omega_n o) - \mathbb{E}[d(o, \omega_n o)]\right)_{n>0}$ converges to $\mathscr{N}(0, \sigma)$ for some $\sigma > 0$.
(2) For each $k > 0$, the RVs $\{\frac{1}{\sqrt{k2^n}} d(o, \omega_{k2^n} o) - \mathbb{E}[d(o, \omega_{k2^n} o)]\}_{n>0}$ are eventually $K/\sqrt[3]{k}$-close to $\mathscr{N}(0, \sigma_k)$.

These two imply that $\mathscr{N}(0, \sigma_k)$ and $\mathscr{N}(0, \sigma)$ are $K/\sqrt[3]{k}$-close. We also have $\mathbb{E}\,|d(o, \omega_n) - \mathbb{E}[d(o, \omega_n)]|^3 \le K' n^{3/2}$ for some $K' > 0$ ([MS20, Theorem 4.9]).

Given n, we fix the following notations throughout the proof:

$$\begin{aligned}
y_i &:= \omega_i o \quad (i = 0, \ldots, n), \\
N_2 &:= \lfloor n^{2/5} \rfloor, \\
N_3 &:= \lfloor n/N_2 \rfloor, \\
Y_{i,n} &:= d(y_{(i-1)N_3}, y_{iN_3}), \quad (i = 1, \ldots, N_2) \\
Y_n^* &:= d(y_{N_2 N_3}, y_n), \\
c^* &:= (o, y_n)_{y_{N_2 N_3}}.
\end{aligned}$$

Next, we define a family of sequences $\{(m(i;k))_{i=0}^{2^k}\}_{k=0}^{\lfloor \log_2 N_2 \rfloor}$ as follows. First we set $m(0;0) = 0$, $m(1;0) = N_2$. Now given $(m(i;k-1))_{i=0}^{2^{k-1}}$ for $k \le \log_2 N_2$, we define $m(2i;k) := m(i;k-1)$ for $i = 0, \ldots, 2^{k-1}$ and

$$m(2i-1;k) := m(i-1;k-1) + \left\lfloor \frac{m(i;k-1) - m(i-1;k-1)}{2} \right\rfloor$$

for $i = 1, \ldots, 2^{k-1}$. Then

$$2^{\lfloor \log_2 N_2 \rfloor - k} \le m(i;k) - m(i-1;k) \le 2^{\lfloor \log_2 N_2 \rfloor - k + 1} \tag{5.6}$$

holds for $k = 0, \ldots, \lfloor \log_2 N_2 \rfloor$ and $i = 1, \ldots, 2^k$.

From this sequences we define

$$b_{i;k} := (y_{N_3 \cdot m(2i-2;k)}, y_{N_3 \cdot m(2i;k)})_{y_{N_3 \cdot m(2i-1;k)}}$$

for $k = 1, \ldots, \lfloor \log_2 N_2 \rfloor - 1$ and $i = 1, \ldots, 2^{k-1}$. Finally, note that

$$\left(m\left(0; \lfloor \log_2 N_2 \rfloor\right), m\left(1; \lfloor \log_2 N_2 \rfloor\right), \ldots, m\left(2^{\lfloor \log_2 N_2 \rfloor}; \lfloor \log_2 N_2 \rfloor\right)\right)$$

is a sequence that increases by 1 or 2 at each step. Let $m'(1) < \ldots < m'(N_2 - 2^{\lfloor \log_2 N_2 \rfloor})$ be the numbers in the sequence that differs with the previous step by 2, and define

$$c_t := (y_{N_3 \cdot m'(t)}, y_{N_3 \cdot m'(t)-2})_{y_{N_3 \cdot m'(t)-1}}.$$

We then observe that

(5.7)

$$\begin{aligned} d(o, \omega_n\, o) &= d(o, \omega_{N_2N_3}\, o) + d(\omega_{N_2N_3}\, o, \omega_n\, o) - 2(o, \omega_n\, o)_{\omega_{N_2N_3}\, o} \\ &= d(o, \omega_{N_2N_3}\, o) + Y_n^* - 2c^* \\ &= \sum_{i=1}^{N_2} Y_{i,n} + 2 \left(\sum_{k=1}^{\lfloor \log_2 N_2 \rfloor} \sum_{i=1}^{2^{k-1}} b_{i;k} \right) + 2 \left(\sum_{i=1}^{N_2 - 2^{\lfloor \log_2 N_2 \rfloor}} c_i \right) + Y_n^* - 2c^*. \end{aligned}$$

For convenience, let us denote by $\bar{Y}$ the centered version $Y - \mathbb{E}[Y]$ of an RV Y. We then also have

(5.8)

$$\begin{aligned} \frac{1}{\sigma\sqrt{n}}[d(o, \omega_n\, o) - \lambda n] = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{N_2} \bar{Y}_{i,n} - \frac{2}{\sigma\sqrt{n}} \left(\sum_{k=1}^{\lfloor \log_2 N_2 \rfloor} \sum_{i=1}^{2^{k-1}} \bar{b}_{i;k} \right) - \frac{2}{\sigma\sqrt{n}} \left(\sum_{i=1}^{N_2 - 2^{\lfloor \log_2 N_2 \rfloor}} \bar{c}_i \right) \\ + \frac{1}{\sigma\sqrt{n}} \bar{Y}_n^* - \frac{2}{\sigma\sqrt{n}} \bar{c}^* + \left(\frac{1}{\sigma\sqrt{n}} \mathbb{E}[d(o, \omega_n\, o)] - \frac{\sqrt{n}\lambda}{\sigma} \right). \end{aligned}$$

We now deal with each term of Equation 5.8. First, note that

$$\mathbb{E}\left[\frac{\sqrt{N_2}}{\sigma\sqrt{n}} \bar{Y}_{i,n}^3 \right] \le K' \frac{N_3^{3/2} N_2^{3/2}}{n^{3/2}} \le K'$$

and

$$\mathbb{E}\left[\frac{\sqrt{N_2}}{\sigma\sqrt{n}} \bar{Y}_{i,n}^2 \right] \ge 0.9 s^2 \frac{N_3 \cdot N_2}{n} \ge 0.8 s^2$$

for large enough n. Then the classical Berry-Esseen estimate asserts that there exists $K > 0$ (that works for all large n) such that

$$|F_n^{(1)}(x) - \mathcal{N}'(x)| \le K \frac{1}{\sqrt[5]{n}}$$

holds for all $x \in \mathbb{R}$, where $F_n^{(1)}(x)$ is the distribution of $\frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{N_2} \bar{Y}_{i,n}$ and $\mathcal{N}'$ is the distribution of $\mathcal{N}\left(0, (\sigma_{N_3}/\sigma) \cdot \sqrt{(N_2N_3)/n}\right)$. Since $\mathcal{N}(0, \sigma_{N_3})$ and $\mathcal{N}(0, \sigma)$ are $K/\sqrt[5]{n}$-close, we have

$$|\mathcal{N}'(x) - \mathcal{N}_1(x)| \le \frac{K}{\sqrt[5]{n}}$$

for all x where $\mathcal{N}_1(x)$ is the distribution of $\mathcal{N}(0, \sqrt{(N_2N_3)/n})$. Moreover, we note $1 - \sqrt{N_2N_3/n} \le K/n^{2/5}$; this implies $|\mathcal{N}'(x) - \mathcal{N}(x)| \le K/\sqrt[5]{n}$ for all x also. Since $\mathcal{N}(x)$ is Lipschitz, it now suffices to show that the remaining terms are $O(1/\sqrt[5]{n})$ outside a set of probability $O(1/\sqrt[5]{n})$.

To deal with the second summation, let us recall that $\{\bar{b}_{i;k}\}_i$ is a family of independent RVs that have uniformly bounded 6th moment. Hence,

$$\mathbb{E}\left[\sum_{i=1}^{2^{k-1}} \bar{b}_{i;k}\right]^6 \le K(2^{k-1})^3$$

for some K that does not depend on k and n. Using the Chebyshev inequality, we have $|\frac{1}{\sigma\sqrt{n}}\sum_i \bar{b}_{i;k}| < n^{-1/5}2^{-k/6}$ outside a set of probability $O(n^{-9/5}2^{4k})$. Summing up these effects, we have

$$\mathbb{P}\left(\frac{2}{\sigma\sqrt{n}}\left(\sum_{k=1}^{\lfloor \log_2 N_2\rfloor}\sum_{i=1}^{2^{k-1}} \bar{b}_{i;k}\right) > \frac{1}{\sqrt[5]{n}}\right) \le 2\cdot 2^{4\log_2 N_2}\cdot O(n^{-9/5}) = O(n^{-1/5}).$$

Similarly, the third term of Equation 5.8 has 6th moment of order $O(n^{-9/5})$ and is bounded by $1/\sqrt[5]{n}$ outside a set of probability $O(n^{-3/5})$. Moreover, the fourth term is a sum of at most N_2 independent RVs with uniformly bounded variance, so its variance is bounded by $O(N_2/n) = O(n^{-3/5})$. Again, it is bounded by $1/\sqrt[5]{n}$ outside a set of probability $O(n^{-1/5})$. The fifth term has variance $O(1/n)$ and can be handled similarly.

Finally, recall the proof in [Cho21a, Section 6] that the error arising from the average, namely, $|\sqrt{n}\lambda - \frac{1}{\sqrt{n}}\mathbb{E}[d(o,\omega_n\, o)]|$, is of order $O(1/\sqrt{n})$. This finishes the proof. $\square$

6. Geodesic tracking

Given a random path $\omega = (\omega_n)_n$ with the set of eventual pivotal times $\mathcal{Q}(\omega) = \{i(1) < i(2) < \ldots\}$, we consider the concatenation $\Gamma = \Gamma(\omega)$ of

$$(\eta_1,\eta_2,\ldots) := ([o,\omega_{i(1)}\, o], [\omega_{i(1)}\, o, \omega_{i(1)+M_0}\, o], [\omega_{i(1)+M_0}\, o, \omega_{i(2)}\, o], [\omega_{i(2)}\, o, \omega_{i(2)+M_0}\, o], \ldots).$$

By Lemma 3.8, Γ is a quasigeodesic. We now show the geodesic tracking with doubled exponent.

Proposition 6.1. *Suppose that μ has finite p-th moment for some $p > 0$. Then for almost every sample path $\omega = (\omega_n)_n$, we have*

$$\lim_{k\to\infty}\frac{d(\omega_k\, o,\Gamma)}{k^{1/2p}} = 0.$$

Proof. By Corollary 5.6, $\min[d(o,\omega_\upsilon\, o), d(o,\check{\omega}_{\check{\upsilon}} o)]^{2p}$ is dominated by an integrable RV. This implies that

$$\sum_k \mathbb{P}\left(\min[d(o,\omega_\upsilon\, o), d(o,\check{\omega}_{\check{\upsilon}} o)] > g(k)\right) < \infty \tag{6.1}$$

for some g such that $\lim_k g(k)/k^{1/2p} = 0$. Note that the probabilities in the summation do not change after the Bernoulli shift T. Note also that $\mathbb{P}(\max\{\upsilon,\check{\upsilon}\} \ge k)$ is summable and is invariant under the Bernoulli shift. By the Borel-Cantelli lemma, we deduce the following for a.e. $(\check{\omega},\omega)$. For each large k, there exists $j = j(k) \in \mathbb{Z}$ such that $|j| \le k$, $d(\omega_k\, o, \omega_{k+j}\, o) \le g(k)$ and either:

(1) there exists $0 < i \le j - M_0$ such that
- $\alpha := (g_{k+i+1}, \ldots, g_{k+i+M_0})$ is a Schottky sequence,
- $(\omega_k\, o, \omega_{k+i}\, \Gamma(\alpha), \omega_{k+n}\, o)$ is D_1-aligned for all $n \ge j$, and
- $(\omega_{k-n'}\, o, \omega_{k+i}\, \Gamma(\alpha))$ is D_2-aligned for all $n' \ge 0$,

or;

(2) there exists $0 > i \ge j + M_0$ such that
- $\alpha := (g_{k+i}^{-1}, g_{k+i-1}^{-1}, \ldots, g_{k+i-M_0+1}^{-1})$ is a Schottky sequence,
- $(\omega_k\, o, \omega_{k+i}\, \Gamma(\alpha), \omega_{k+n}\, o)$ is D_1-aligned for all $n \le j$,
- $(\omega_{n'+k}\, o, \omega_{k+i}\, \Gamma(\alpha))$ is D_2-aligned for all $n' \ge 0$.

The first case is where j equals $\upsilon(T^k(\check{\omega}, \omega))$ and the second case is where j equals $-\check{\upsilon}(T^k(\check{\omega}, \omega))$. In both cases, the second item for $n = j$ leads to

$$d(\omega_k\, o, \omega_{k+i}\, \Gamma(\alpha)) \le d(\omega_k\, o, \omega_{k+j}\, o) \le g(k).$$

We now let $N = k + |j|$; note $i(N) > N$. In the first case of the dichotomy, $(o, \omega_{k+i}\, \Gamma(\alpha), \omega_{i(N)}\, o)$ is D_2-aligned. In the second case, $(\omega_{i(N)}\, o, \omega_{k+i}\, \Gamma(\alpha), o)$ is D_2-aligned. We now claim that $d(\eta_m, \omega_{k+i}\, \Gamma(\alpha))$ is bounded for some m.

The projections of the beginning point of η_1 and the terminating point of η_{2N-1} onto $\omega_{k+i}\, \Gamma(\alpha)$ are far away. Hence, one of the following holds.

(a) some η_m has a large projection on $\omega_{k+i}\, \Gamma(\alpha)$: more precisely, there exists η_m with endpoints $\{x_m, y_m\}$ such that

$$\begin{aligned} d(\pi_{\omega_{k+i}\, \Gamma(\alpha)}(x_m), \omega_{k+i}\, o) &\le 2K_0 + K_3 + 2E_0 + D_2, \\ d(\pi_{\omega_{k+i}\, \Gamma(\alpha)}(y_m), \omega_{k+i+M_0}\, o) &\le 2K_0 + K_3 + 2E_0 + D_2, \end{aligned}$$

(b) an endpoint p of some η_m projects onto $\omega_{k+i}\, \Gamma(\alpha)$ in the middle, i.e.,

$$d(\pi_{\omega_{k+i}\, \Gamma(\alpha)}(p), \omega_{k+i}\, o), d(\pi_{\omega_{k+i}\, \Gamma(\alpha)}(p), \omega_{k+i+M_0}\, o) > 2K_0 + K_3 + 2E_0 + D_2.$$

Recall that

$$d(\omega_{k+i}\, o, \omega_{k+i+M_0}\, o) \ge 2\left(\frac{M_0}{K_0} - K_0\right) \ge 6K_0 + 2K_3 + 4E_0 + 2D_2.$$

Hence, in Case (a), we deduce $d\left(\pi_{\omega_{k+i}\, \Gamma(\alpha)}(x_i), \pi_{\omega_{k+i}\, \Gamma(\alpha)}(y_i)\right) \ge 2K_0$ and η_i is within a neighborhood of $\omega_{k+i}\, \Gamma(\alpha)$ by the K_0-BGIP of $\Gamma(\alpha)$.

In Case (b), recall that the Schottky axes at eventual pivotal times are parts of a D_0-aligned sequence; by Proposition 3.6, p is within d-distance E_1 from some $q \in [o, \omega_{n_t}\, o]$. Then q also projects onto $\omega_{k+i}\, \Gamma(\alpha)$ in the middle:

$$d(\pi_{\omega_{k+i}\, \Gamma(\alpha)}(q), \omega_{k+i}\, o), d(\pi_{\omega_{k+i}\, \Gamma(\alpha)}(q), \omega_{k+i+M_0}\, o) > 2K_0 + D_2.$$

Since the projections of $[o, q]$ and $[q, \omega_{n_t}\, o]$ onto $\omega_{k+i}\, \Gamma(\alpha)$ are both large, we can apply Lemma 2.5 and obtain $q_1 \in [o, q]$, $q_2 \in [q, \omega_{n_t}\, o]$ such that $d(q_1, \pi_{\omega_{k+i}\, \Gamma(\alpha)}(q)), d(q_2, \pi_{\omega_{k+i}\, \Gamma(\alpha)}(q)) < K_3$. This forces that p is also near $\omega_{k+i}\, \Gamma(\alpha)$. $\square$

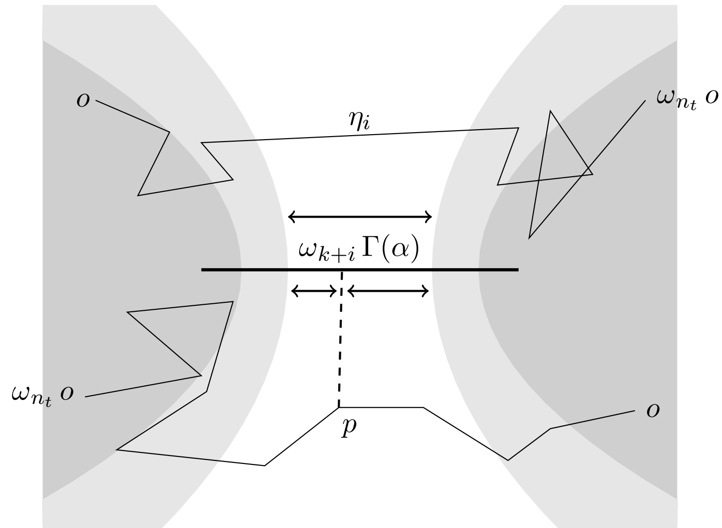

FIGURE 10. Dichotomy in the proof of Proposition 6.1. o and $\omega_{n_t}\, o$ are distant when seen from $\omega_{k+i}\,\Gamma(\alpha)$, so either an η_i is seen large (the upper case) or an endpoint p of some η_i is seen in the middle (the lower case).

In the previous lemma, we only assumed $p > 0$. Namely, sublinear tracking occurs even when μ, $\check{\mu}$ has finite (1/2)-th moment only. When μ has finite exponential moment, the exact same proof works with $g(k) = C \log k$ for some suitable C. This leads to the following:

Proposition 6.2. *Suppose that μ has finite exponential moment. Then there exists $C > 0$ such that for almost every sample path $\omega = (\omega_n)_n$, we have*

$$\limsup_{k\to\infty} \frac{d(\omega_k\, o, \Gamma)}{\log k} \le C.$$

APPENDIX A. PROOFS OF LEMMATA FOR THE SET OF PIVOTAL TIMES

In this section, we provide proofs for Lemma 4.1, 4.3 and 4.4.

We begin with the following consequence of Lemma 3.18 and Lemma 3.3:

Observation A.1. *For any $s \in S^{4n}$ and $1 \le i \le n$, $(\Upsilon(\alpha_i), \Upsilon(\beta_i))$ and $(\Upsilon(\gamma_i), \Upsilon(\delta_i))$ are D_0-aligned.*

Lemma A.2 ([Cho21b, Lemma 3.1])**.** *Let $l < m$ be consecutive elements in P_k, i.e., $l, m \in P_k$ and $l = \max(P_k \cap \{1, \ldots, m-1\})$. Then there exists a sequence $\{l = i(1) < \ldots < i(M) = m\} \subseteq P_k$ with cardinality $M \ge 2$ such that*

$$\big(\Upsilon(\delta_l), \Upsilon(\alpha_{i(2)}), \Upsilon(\beta_{i(2)}), \ldots, \Upsilon(\alpha_{i(M-1)}), \Upsilon(\beta_{i(M-1)}), \Upsilon(\alpha_m)\big)$$

is D_0-aligned.

Proof. $l, m \in P_n$ implies that $l \in P_l$ and $l, m \in P_m$. In particular, l (m, resp.) is newly chosen at step l (m, resp.) by fulfilling Criterion (A). Hence, $(\Upsilon(\delta_l), y^-_{l+1,2})$ and $(z_{m-1}, \Upsilon(\alpha_m))$ are K_0-aligned $(*)$, and $z_l = y^+_{l,1}$. Moreover, we have $P_m = P_{m-1} \cup \{m\}$ and $l = \max P_{m-1}$.

If $l = m-1$ and m was newly chosen at step $m = l+1$, then $z_{m-1} = z_l = y^+_{l,1}$ holds. Then Lemma 3.3 and $(*)$ imply that $(\Upsilon(\delta_l), \Upsilon(\alpha_m))$ is D_0-aligned.

If $l < m-1$, then $l = \max P_{m-1}$ has survived at step $m-1$ by fulfilling Criterion (B); there exist $l = i(1) < \ldots < i(M-1)$ in P_{m-2} (with $M-1 \geq 2$) such that:

- $\big(\Upsilon(\delta_{i(1)}), \Upsilon(\alpha_{i(2)}), \Upsilon(\beta_{i(2)}), \ldots, \Upsilon(\alpha_{i(M-1)}), \Upsilon(\beta_{i(M-1)})\big)$ is D_0-aligned;
- $\Big(\Upsilon(\beta_{i(M-1)}), y^-_{n+1,2}\Big)$ is K_0-aligned, and
- z_{m-1} equals $y^-_{i(M-1),1}$, the beginning point of $\Upsilon(\beta_{i(M-1)})$.

We have also observed that $(z_{m-1}, \Upsilon(\alpha_m))$ is K_0-aligned $(*)$. Then Lemma 3.3 asserts that $\big(\Upsilon(\beta_{i(M-1)}), \Upsilon(\alpha_m)\big)$ is D_0-aligned as desired. □

Let us now prove Lemma 4.1.

Proof. Considering the previous lemma, it suffices to prove the following:

- $\big(o, \Upsilon(\alpha_{i(1)})\big)$ is K_0-aligned;
- for each $1 \leq t \leq m$, $\big(\Upsilon(\alpha_{i(t)}), \Upsilon(\beta_{i(t)}), \Upsilon(\gamma_{i(t)}), \Upsilon(\delta_{i(t)})\big)$ is D_0-aligned;
- there exist finitely many Schottky axes $\Upsilon(\delta_{i(m)}) = \Upsilon_1, \ldots, \Upsilon_M$ such that $\Big(\Upsilon_1, \ldots, \Upsilon_M, y^-_{n+1,2}\Big)$ is D_0-aligned.

Note that for each $t = 1, \ldots, m$, $i(t)$ is newly chosen as a pivotal time at step $i(t)$ by fulfilling Criterion (A). In particular, we have that:

- $(\Upsilon(\alpha_n), \Upsilon(\beta_n))$ is D_0-aligned (Observation A.1);
- $(\Upsilon(\beta_n), \Upsilon(\gamma_n))$ is D_0-aligned since $\Big(\Upsilon(\beta_n), y^+_{n,1}\Big)$ and $\Big(y^-_{n,0}, \Upsilon(\gamma_n)\Big)$ are K_0-aligned (Lemma 3.3), and
- $(\Upsilon(\gamma_n), \Upsilon(\delta_n))$ is D_0-aligned (Observation A.1).

This guarantees the second item.

We also note that $P_{i(1)-1} = \emptyset$. Indeed, any j in $P_{i(1)-1}$ is smaller than $i(1)$ and would have survived in $P_{i(1)}$ (since what happened at step $i(1)$ was an addition of an element, not a deletion). Since $i(1)$ was not deleted at any later step, such j would also not be deleted till the end and should have appeared in P_n. Since $i(1)$ is the earliest pivotal time in P_n, no such j exists. Hence, $z_{i(1)-1} = o$ and Criterion (A) for $i(1)$ leads to the first item.

We now observe how $i(m)$ survived in P_n. If $i(m) = n$, then it was newly chosen at step n by fulfilling Criterion (A). In particular, $(\Upsilon(\delta_n), y^-_{n+1,2})$ is K_0-aligned as desired.

If $i(m) \neq n$, then it has survived at step n as the last pivotal time by fulfilling Criterion (B). In particular, there exist $\{i(m) = j(1) < \ldots <$

$j(k)\} \subseteq P_{n-1}$ $(k > 1)$ such that

$$(\kappa_i)_{i=1}^{2k-1} = \big(\Upsilon(\delta_{j(1)}), \Upsilon(\alpha_{j(2)}), \Upsilon(\beta_{j(2)}), \ldots, \Upsilon(\alpha_{j(k)}), \Upsilon(\beta_{j(k)})\big)$$

is D_0-aligned and $\left(\Upsilon(\beta_{j(k)}), y^{-}_{n+1,2}\right)$ is K_0-aligned. □

Next, we prove Lemma 4.3.

Proof. Since $\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}, \delta_{i-1}$ are intact, $P_l(s) = P_l(\bar{s})$ and $\tilde{S}'_l(s) = \tilde{S}'_l(\bar{s})$ hold for $l = 0, \ldots, i-1$. At step i, δ_i satisfies Condition 4.3 (since $i \in P_k(s)$) and $(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i)$ satisfies Condition 4.2, 4.4 and 4.5. Hence, i is newly added in $P_i(\bar{s})$ and

$$P_i(\bar{s}) = P_{i-1}(\bar{s}) \cup \{i\} = P_{i-1}(s) \cup \{i\} = P_i(s).$$

We also have $\tilde{S}_i(s) = \tilde{S}_i(\bar{s})$ as z_{i-1}, $w^{-}_{i,2}$ are not affected. Meanwhile, z_i is modified into $\bar{z}_i = \bar{y}^{+}_{i,1} = gy^{+}_{i,1} = gz_i$, where $g := w^{-}_{i,2}\bar{a}_i\bar{b}_i v_i \bar{c}_i (w^{-}_{i,2}a_i b_i v_i c_i)^{-1}$. More generally, we have

$$\begin{aligned} w^{-}_{l,t} &= gw^{-}_{l,t} \quad (t \in \{0,1,2\},\ l > i), \\ w^{+}_{l,0} &= gw^{+}_{l,0} \quad (l > i), \\ w^{+}_{l,t} &= gw^{+}_{l,t} \quad (t \in \{1,2\}, l \geq i). \end{aligned} \tag{A.1}$$

We now claim the following for $i < l \leq k$:

(1) If s fulfills Criterion (A) at step l, then so does $\bar{s}$.
(2) If not and $\{i(1) < \ldots < i(M)\} \subseteq P_{l-1}(s)$ is the maximal sequence for s in Criterion (B) at step l, then it is also the maximal one for $\bar{s}$ at step l.
(3) In both cases, we have $P_l(s) = P_l(\bar{s})$ and $\bar{z}_l = gz_l$.

Assuming the third item for $l-1$: $P_{l-1}(s) = P_{l-1}(\bar{s})$ and $\bar{z}_{l-1} = gz_{l-1}$, Equality A.1 implies the first item. In this case we also deduce $P_l(s) = P_{l-1}(s) \cup \{l\} = P_{l-1}(\bar{s}) \cup \{l\} = P_l(\bar{s})$ and $\bar{z}_l = \bar{y}^{+}_{l,1} = gy^{+}_{l,1} = gz_l$, the third item for l.

Furthermore, Equality A.1 implies that a sequence $\{i(1) < \ldots < i(M)\}$ in $P_{l-1}(s) \cap \{i, \ldots, l-1\} = P_{l-1}(\bar{s}) \cap \{i, \ldots, l-1\}$ works for s in Criterion (B) if and only if it works for $\bar{s}$. Note that $i \in P_l(s)$ since $i \in P_k(s)$ and $l \leq k$; hence, such sequences exist and the maximal sequence is chosen among them. Therefore, the maximal sequence $\{i(1) < \ldots < i(M)\}$ for s is also maximal for $\bar{s}$. We then deduce $P_l(s) = P_{l-1}(s) \cap \{1, \ldots, i(1)\} = P_{l-1}(\bar{s}) \cap \{1, \ldots, i(1)\} = P_l(\bar{s})$ and $\bar{z}_l = \bar{y}^{-}_{i(M),1} = gy^{-}_{i(M),1} = gz_l$ (noting that $i(M) > i$), the third item for l.

Since we have $\bar{z}_i = gz_i$, induction shows that $P_l(s) = P_l(\bar{s})$ for each $i < l \leq k$. Moreover, Equality A.1 and $\bar{z}_{l-1} = gz_{l-1}$ imply that $\tilde{S}_l(s) = \tilde{S}_l(\bar{s})$. □

We finally prove Lemma 4.4. Recall that $\mathcal{E}_j(s)$ is the set of choices pivoted from the choice $s \in S^{4j}$. Also recall that being pivoted from each other is an equivalence relation, by Lemma 4.3.

Proof. Let us fix $s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}, \delta_{k-1}) \in S^{4(k-1)}$ and

$$\mathcal{A} := \Big\{(\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4 : \#P_k(s, \alpha_k, \beta_k, \gamma_k, \delta_k) = \#P_{k-1}(s) + 1\Big\}.$$

Then Lemma 4.2 implies that $\mathbb{P}(\mathcal{A} \,|\, S^4) \geq 1 - 4/N_0$. Moreover, for $(\alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}$ we have $P_{k-1}(s) \subseteq P_{k-1}(s) \cup \{k\} = P_k(s, \alpha_k, \beta_k, \gamma_k, \delta_k)$. Hence, $(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k)$ is pivoted from $(s, \alpha_k, \beta_k, \gamma_k, \delta_k)$ for any $\tilde{s} \in \mathcal{E}_{k-1}(s)$. Lemma 4.3 then implies that $P_k(\tilde{s}) = P_k(s) = P_{k-1}(s) \cup \{k\} = P_{k-1}(\tilde{s}) \cup \{k\}$, and we have

$$\begin{aligned}&\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(\tilde{s}) \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s), (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right) \\ &\leq 1 - \mathbb{P}(\mathcal{A}|S^4) \leq 4/N_0.\end{aligned}$$

This settles the case $j = 0$.

Now let $j = 1$. The event under discussion becomes void when $\#P_{k-1}(s) < 2$. Excluding such cases, let $l < m$ be the last 2 elements of $P_{k-1}(s)$. For each $\tilde{s} \in \mathcal{E}_{k-1}(s)$ and $A \subseteq S^3$ we define

$$E(\tilde{s}, A) := \left\{ \bar{s} = (\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i, \bar{\delta}_i)_{i=1}^{k-1} : \begin{array}{c} \bar{\alpha}_i = \tilde{\alpha}_i,\ \bar{\gamma}_i = \tilde{\gamma}_i,\ \bar{\delta}_i = \tilde{\delta}_i \text{ for all } i, \\ \bar{\beta}_i = \tilde{\beta}_i \text{ for } i \neq m, \\ (\tilde{\alpha}_m, \bar{\beta}_m, \tilde{\gamma}_m) \in A \end{array} \right\}.$$

In other words, we only modify a single choice of $\tilde{\beta}_m$ in a way that the modified triple at step m belongs to A. Then $\{E(\tilde{s}, \tilde{S}_m(s)) : \tilde{s} \in \mathcal{E}_{k-1}(s)\}$ partitions $\mathcal{E}_{k-1}(s)$ by Lemma 4.3. Note that for each $\tilde{s} \in \mathcal{E}_{k-1}(s)$, the size of $E(\tilde{s}, \tilde{S}_m(s))$ is the number of $\bar{\beta}_m \in S$ that satisfies Condition 4.4 (with $\tilde{\gamma}_m$ instead of γ_m there); there are at least $\#S - 1$ such choices.

We now fix $(\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4$ and $\tilde{s} = (\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i, \tilde{\delta}_i)_{i=1}^{k-1} \in \mathcal{E}_{k-1}(s)$. Let $\tilde{A} = \tilde{A}(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \subseteq \tilde{S}_m(s)$ be the collection of elements $(\tilde{\alpha}_m, \bar{\beta}_m, \tilde{\gamma}_m)$ in $\tilde{S}_m(s)$ such that $\bar{\beta}_m$ satisfies

(A.2)
$$\begin{aligned}&\operatorname{diam}\left(\pi_{\Gamma^{-1}(\bar{\beta}_m)}((\tilde{w}_{m,0}^-)^{-1}\tilde{w}_{k-1,2}^- a_k b_k v_k c_k d_k o) \cup o\right) \\ &= \operatorname{diam}\left(o \cup \pi_{\Gamma^{-1}(\bar{\beta}_m)}(v_m \tilde{c}_m \tilde{d}_m w_m \cdots \tilde{a}_{k-1}\tilde{b}_{k-1} v_{k-1} \tilde{c}_{k-1}\tilde{d}_{k-1} w_{k-1} \cdot a_k b_k v_k c_k d_k w_k o)\right) < K_0.\end{aligned}$$

The size of $\tilde{A}$ is the number of $\bar{\beta}_m \in S$ that satisfies Condition 4.4 plus Condition A.2; there are at least $\#S - 2$ such choices.

We claim that $\#P_k(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \geq \#P_{k-1}(s) - 1$ for $\bar{s} \in E(\tilde{s}, \tilde{A})$. First, since $l < m$ are consecutive elements in $P_{k-1}(\bar{s})$, Lemma A.2 gives a sequence $\{l = i(1) < \ldots < i(M) = m\} \subseteq P_{k-1}$ such that

$$\big(\Upsilon(\bar{\delta}_{i(1)}), \Upsilon(\bar{\alpha}_{i(2)}), \Upsilon(\bar{\beta}_{i(2)}), \ldots, \Upsilon(\bar{\alpha}_{i(M-1)}), \Upsilon(\bar{\beta}_{i(M-1)}), \Upsilon(\bar{\alpha}_m)\big)$$

is D_0-aligned. Moreover, Observation A.1 and Condition A.2 imply that

$$\big(\Upsilon(\bar{\alpha}_m), \Upsilon(\bar{\beta}_m)\big), \quad \left(\Upsilon(\bar{\beta}_m), \bar{y}_{k+1,2}^-\right)$$

are D_0-aligned and K_0-aligned, respectively. In summary, $\{l = i(1) < \cdots < i(M)\} \subseteq P_{k-1}(\bar{s})$ works for $(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k)$ in Criterion (B) at step k. This implies $P_k(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \supseteq P_{k-1}(\bar{s}) \cap \{1, \ldots, l\}$, hence the claim.

As a result, for each $\tilde{s} \in \mathcal{E}_{k-1}(s)$ we have

$$\begin{aligned}&\mathbb{P}\left(\#P_k(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - 1 \,\middle|\, \bar{s} \in E(\tilde{s}, \tilde{S}_m)\right) \\ &\le \frac{\#\left[E(\tilde{s}, \tilde{S}_m) \setminus E(\tilde{s}, \tilde{A})\right]}{\#E(\tilde{s}, \tilde{S}_m)} \le \frac{3}{\#S - 1} \le \frac{4}{N_0}.\end{aligned}$$

Since $E(\tilde{s}, \bar{S}_m)$'s for $\tilde{s} \in \mathcal{E}_{k-1}(s)$ partition $\mathcal{E}_{k-1}(s)$, we deduce

$$\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - 1 \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s)\right) \le \frac{4}{N_0}.$$

Moreover, the above probability vanishes when $(\alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}$. Since $\mathbb{P}(\mathcal{A} \,|\, S^4) \ge 1 - 4/N_0$, we deduce that

(A.3)
$$\begin{aligned}&\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - 1 \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s), (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right) \\ &\le \frac{4}{N_0} \cdot \frac{4}{N_0} \le \left(\frac{4}{N_0}\right)^2.\end{aligned}$$

Now let $j = 2$. We similarly discuss only for s such that $\#P_{k-1}(s) \ge 3$; let $l' < l < m$ be the last 3 elements. For $(\bar{\alpha}_m, \bar{\beta}_m, \bar{\gamma}_m) \in S^3$ we define

$$s'(\bar{\alpha}_m, \bar{\beta}_m, \bar{\gamma}_m) := (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \bar{\alpha}_m, \bar{\beta}_m, \bar{\gamma}_m, \delta_m, \ldots, \alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}, \delta_{k-1}).$$

In other words, $s'(\bar{\alpha}_m, \bar{\beta}_m, \bar{\gamma}_m)$ is obtained from s by replacing α_m with $\bar{\alpha}_m$, β_m with $\bar{\beta}_m$ and γ_m with $\bar{\gamma}_m$. We then define

$$\mathcal{A}_1 := \left\{ \begin{pmatrix} \bar{\alpha}_m, \bar{\beta}_m, \bar{\gamma}_m, \\ \alpha_k, \beta_k, \gamma_k, \delta_k \end{pmatrix} \in \tilde{S}_m(s) \times S^4 : \#P_k\left(s'(\bar{\alpha}_m, \bar{\beta}_m, \bar{\gamma}_m), \alpha_k, \beta_k, \gamma_k, \delta_k\right) \ge \#P_{k-1}(s) - 1 \right\}.$$

Equivalently, we are requiring

$$P_{k-1}(s) \cap \{1, \ldots, l\} \subseteq P_k\left(s', \alpha_k, \beta_k, \gamma_k, \delta_k\right).$$

(This equivalence relies on the fact $P_{k-1}(s') = P_{k-1}(s)$ due to Lemma 4.3.)

Observation A.3. *Let*

$$\tilde{s} = (\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i, \tilde{\delta}_i)_{i=1}^{k-1} \in \mathcal{E}_{k-1}(s), \quad (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4.$$

Then $(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}_1$ *if and only if* $\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \ge \#P_{k-1}(s) - 1$.

To see this, suppose first that $(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}_1$. Then $(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k)$ is pivoted from $\left(s'(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m), \alpha_k, \beta_k, \gamma_k, \delta_k\right)$, as the former choice differs from the latter choice only at entries $(\alpha_i, \beta_i, \gamma_i)$'s for $i \in P_{k-1}(s) \cap \{1, \ldots, l\} \subseteq P_k\left(s', \alpha_k, \beta_k, \gamma_k, \delta_k\right)$. Lemma 4.3 then implies that

$$P_{k-1}(s) \cap \{1, \ldots, l\} \subseteq P_k(s', \alpha_k, \beta_k, \gamma_k, \delta_k) = P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k)$$

and $\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \geq \#P_{k-1}(s) - 1$.

Conversely, suppose $\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \geq \#P_{k-1}(s) - 1$. This amounts to saying

$$P_{k-1}(s) \cap \{1, \ldots, l\} \subseteq P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k).$$

Then $\left(s'(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m), \alpha_k, \beta_k, \gamma_k, \delta_k\right)$ is pivoted from $(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k)$, as the former choice differs from the latter choice only at entries $(\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i)$'s for $i \in P_{k-1}(s) \cap \{1, \ldots, l\} \subseteq P_k\left(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k\right)$. Lemma 4.3 then implies that

$$P_{k-1}(s) \cap \{1, \ldots, l\} \subseteq P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) = P_k(s', \alpha_k, \beta_k, \gamma_k, \delta_k)$$

and $(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}_1$.

Combining Observation A.3 and Inequality A.3, we deduce

$$\begin{aligned}
&\mathbb{P}(\mathcal{A}_1 \,|\, \tilde{S}'_m(s) \times S^4) \\
&= \mathbb{P}\left((\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}_1 \,|\, \tilde{s} \in \mathcal{E}_{k-1}(s), (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right) \\
&= \mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \geq \#P_{k-1}(s) - 1 \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s), (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right) \\
&\geq 1 - \left(\frac{4}{N_0}\right)^2.
\end{aligned}$$

We now define for $\tilde{s} \in \mathcal{E}_{k-1}(s)$ and $A \subseteq S^3$

$$E_1(\tilde{s}, A) := \left\{ \bar{s} = (\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i, \bar{\delta}_i)_{i=1}^{k-1} : \begin{array}{c} \bar{\alpha}_i = \tilde{\alpha}_i,\ \bar{\gamma}_i = \tilde{\gamma}_i,\ \bar{\delta}_i = \tilde{\delta}_i \text{ for all } i, \\ \bar{\beta}_i = \tilde{\beta}_i \text{ for } i \neq l, \\ (\tilde{\alpha}_l, \bar{\beta}_l, \tilde{\gamma}_l) \in A \end{array} \right\}.$$

Then $\{E_1(\tilde{s}, \tilde{S}_l(s)) : \tilde{s} \in \mathcal{E}_{k-1}(s)\}$ partitions $\mathcal{E}_{k-1}(s)$ by Lemma 4.3. Moreover, for each $\tilde{s} \in \mathcal{E}_{k-1}(s)$ we have $\#E(\tilde{s}, \tilde{S}_l(s)) \geq \#S - 1$.

Now fixing $(\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4$ and $\tilde{s} = (\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i, \tilde{\delta}_i)_{i=1}^{k-1} \in \mathcal{E}_{k-1}(s)$, let $\tilde{A}_1 = \tilde{A}_1(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \subseteq \tilde{S}_l(s)$ be the collection of elements $(\tilde{\alpha}_l, \bar{\beta}_l, \tilde{\gamma}_l)$ that satisfies

$$\text{diam}\left(\pi_{\Gamma^{-1}(\bar{\beta}_l)}((\tilde{w}_{l,0}^-)^{-1}\tilde{w}_{k-1,2}^- a_k b_k v_k c_k d_k o) \cup o\right) < K_0. \tag{A.4}$$

As before, the size of $\tilde{A}_1$ is at least $\#S - 2$.

We now claim that $\#P_k(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \geq \#P_{k-1}(s) - 2$ for $\bar{s} \in E_1(\tilde{s}, \tilde{A}_1)$. First, since $l' < l$ are consecutive elements in $P_{k-1}(\bar{s})$, Lemma A.2 gives a sequence $\{l' = i(1) < \ldots < i(M) = l\} \subseteq P_{k-1}$ such that

$$\left(\Upsilon(\bar{\delta}_{i(1)}), \Upsilon(\bar{\alpha}_{i(2)}), \Upsilon(\bar{\beta}_{i(2)}), \ldots, \Upsilon(\bar{\alpha}_{i(M-1)}), \Upsilon(\bar{\beta}_{i(M-1)}), \Upsilon(\bar{\alpha}_l)\right)$$

is D_0-aligned. Moreover, Observation A.1 and Condition A.2 imply that

$$\left(\Upsilon(\bar{\alpha}_l), \Upsilon(\bar{\beta}_l)\right), \quad \left(\Upsilon(\bar{\beta}_l), \bar{y}_{k+1,2}^-\right)$$

is D_0-aligned and K_0-aligned, respectively. In summary, $\{l' = i(1) < \cdots < i(M)\} \subseteq P_{k-1}(\bar{s})$ works for $(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k)$ in Criterion (B) at step k. This implies $P_k(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \supseteq P_{k-1}(\bar{s}) \cap \{1, \ldots, l'\}$, hence the claim.

As a result, for each $\tilde{s} \in \mathcal{E}_{k-1}(s)$ we have

$$\mathbb{P}\left(\#P_k(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - 2 \,\middle|\, \bar{s} \in E_1(\tilde{s}, \tilde{S}_l)\right)$$
$$\le \frac{\#\left[E(\tilde{s}, \tilde{S}_l) \setminus E(\tilde{s}, \tilde{A}_1)\right]}{\#E(\tilde{s}, \tilde{S}_l')} \le \frac{3}{\#S - 1} \le \frac{4}{N_0}.$$

Moreover, Observation A.3 asserts that the above probability vanishes for $\tilde{s}$ and $(\alpha_k, \beta_k, \gamma_k, \delta_k)$ such that $(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}_1$. Since

$$\mathbb{P}\left[\bigcup \{E_1(\tilde{s}, \tilde{S}_l) \times (\alpha_k, \beta_k, \gamma_k, \delta_k) : (\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \notin \mathcal{A}_1\} \,\middle|\, \mathcal{E}_{k-1}(s) \times S^4\right]$$
$$= \mathbb{P}\left[(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \notin \mathcal{A}_1 \,\middle|\, \tilde{S}_m(s) \times S^4\right] \le (4/N_0)^2,$$

we sum up the conditional probabilities to obtain
(A.5)

$$\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - 2 \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s), (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right)$$
$$\le \left(\frac{4}{N_0}\right)^2 \times \frac{4}{N_0} \le \left(\frac{4}{N_0}\right)^3.$$

We repeat this procedure to cover all $j < \#P_{k-1}(s)$. The case $j \ge \#P_{k-1}(s)$ is void. $\square$

Lemma 4.3 leads to the estimation in Corollary 4.5.

Proof. Let $\{X_i\}_i$ be the family of i.i.d. as in Equation 4.6 that is also assumed to be independent from the choice s. Lemma 4.2 and Lemma 4.4 imply the following: for $0 \le k < n$ and any i,

$$\mathbb{P}\left(\#P_{k+1}(s) \ge i + j \,\middle|\, \#P_k(s) = i\right) \ge \begin{cases} 1 - \frac{4}{N_0} & \text{if } j = 1, \\ 1 - \left(\frac{4}{N_0}\right)^{-j+1} & \text{if } j < 0. \end{cases} \tag{A.6}$$

Hence, there exists a nonnegative RV U_k such that $\#P_{k+1} - U_k$ and $\#P_k + X_{k+1}$ have the same distribution.

For each $1 \le k \le n$, we claim that $\mathbb{P}(\#P_k \ge i) \ge \mathbb{P}(X_1 + \cdots + X_k \ge i)$ for each i. For $k = 1$, we have $\#P_{k-1} = 0$ always and the claim follows from Inequality A.6. Given the claim for k, we have

$$\begin{aligned}
\mathbb{P}(\#P_{k+1} \ge i) \ge \mathbb{P}(\#P_k + X_{k+1} \ge i) &= \sum_j \mathbb{P}(\#P_k \ge j)\, \mathbb{P}(X_{k+1} = i - j) \\
&\ge \sum_j \mathbb{P}(X_1 + \cdots + X_k \ge j)\, \mathbb{P}(X_{k+1} = i - j) \\
&= \mathbb{P}(X_1 + \cdots + X_k + X_{k+1} \ge i).
\end{aligned}$$

The second assertion follows from a similar induction on $\{\#P_{k+l} - \#P_k\}_{l \ge 0}$.

The final assertion holds since X_i's have finite exponential moments and expectation greater than $1 - 9/N_0$. $\square$

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