RANDOM WALKS AND CONTRACTING ELEMENTS II: TRANSLATION LENGTH AND QUASI-ISOMETRIC EMBEDDING

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Abstract. Continuing from [Cho22b], we study random walks on metric spaces with contracting elements. We prove that random subgroups of the isometry group of a metric space is quasi-isometrically embedded into the space. We discuss this problem in two senses, namely, one involving random walks and the other involving counting problems. We also establish the genericity of contracting elements and the CLT and its converse for translation length.

Keywords. Random walk, Outer space, Teichmüller space, CAT(0) space, Central limit theorem, Contracting property, Quasi-isometric embedding

MSC classes: 20F67, 30F60, 57M60, 60G50

Contents

1. Introduction

This is the second in a series of articles concerning random walks on metric spaces with contracting elements. This series is a reformulation of the previous preprint [Cho22a] announced by the author, aiming for clearer and more concise expositions. Our aim is to provide a unified approach to various limit laws for random walks under optimal moment conditions. Let us recall the setting:

Convention 1.1. *Throughout, we assume that:*

- *(X, d) is a geodesic metric space;*
- *G is a countable group of isometries of X, and*
- *G contains two independent contracting isometries.*

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We also fix a basepoint $o \in X$.

We emphasize that this setting embraces Teichmüller space, Gromov hyperbolic spaces, CAT(0) spaces and many more. In this article, we focus on the asymptotic description of the translation length $\tau(\omega_n)$ of a random isometry ω_n. Our main results are as follows.

Theorem A. *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. Let $\lambda(\omega)$ be the escape rate of ω. Then for any $0 < L < \lambda(\omega)$, there exists $K > 0$ such that*

$$\mathbb{P}\left(\omega_n \textit{ is contracting and } \tau(\omega_n) \geq Ln\right) \geq 1 - Ke^{-n/K}$$

holds.

This result has been observed by Sisto for simple random walks on various spaces in [Sis18]. In the absence of moment conditions, Maher and Tiozzo observed in [MT18] that non-elementary random walks on Gromov hyperbolic spaces favor loxodromic elements in probability. Their methodology and Benoist-Quint's estimates in [BQ16a] also lead to the stronger SLLN for translation length under finite second moment condition, as noted by Dahmani and Horbez [DH18]. Dahmani and Horbez also deduced the same SLLN on Teichmüller space. Later, Baik, Choi and Kim obtained the same SLLN under finite first moment assumption using ergodic theorems and Maher-Tiozzo's notion of persistent joint [BCK21]. We also note Le Bars' recent result [LB22] that non-elementary random walks on a proper CAT(0) space favor contracting isometries in probability. Theorem A generalizes all of the aforementioned results by obtaining an exponential bound from below without any moment condition.

The genericity of contracting elements is a recurring propaganda that also appears in other settings. For example, Yang describes the genericity of contracting elements in counting problem for proper actions on a metric space [Yan20]. Our aim here is to deduce the genericity of contracting elements for possibly non-WPD actions on spaces and for non-elementary random walks without moment condition.

We also provide a quantitative comparison between the displacement and the translation length of a random isometry.

Theorem B. *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G.*

(1) If μ has finite p-th moment for some $p > 0$, then

$$\lim_{n \to \infty} \frac{1}{n^{1/2p}} [d(o, \omega_n\, o) - \tau(\omega_n)] = 0 \quad a.s.$$

(2) If μ has finite first moment, then there exists $K > 0$ such that

$$\limsup_{n \to \infty} \frac{1}{\log n} [d(o, \omega_n\, o) - \tau(\omega_n)] \leq K \quad a.s.$$

There has been observations that displacement and translation length have sublinear discrepancy for random walks with bounded support or with finite exponential moment: see [Mah12], [MT18]. Moreover, using Benoist-Quint's strategy in [BQ16a], [BQ16b] and its application to other spaces ([Hor18], [DH18]), one can achieve sublinear discrepancy for random walks with finite first moment. We improve these observations by proving that random walks with finite (1/2)-th moment exhibit sublinear discrepancy between displacement and translation length.

These theorems lead to the SLLN and CLT for translation length. In particular, we complete the CLT in [Cho22b] as follows:

Theorem C (CLT and its converse)**.** *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. If μ has finite second moment, then $\frac{1}{\sqrt{n}}(d(o, \omega_n\, o) - n\lambda)$ and $\frac{1}{\sqrt{n}}(\tau(\omega_n) - n\lambda)$ converge to the same Gaussian distribution in law.*

Conversely, suppose that μ has infinite second moment. Then for any sequence $(c_n)_n$, both $\frac{1}{\sqrt{n}}(d(o, \omega_n\, o) - c_n)$ and $\frac{1}{\sqrt{n}}(\tau(\omega_n) - c_n)$ do not converge in law.

CLT for translation length and the converses of CLT for Gromov hyperbolic spaces and Teichmüller space have been observed in [Cho21a]. Here, we establish the same result for general spaces with contracting isometries.

Meanwhile, Taylor and Tiozzo proved in [TT16] that random subgroups of a weakly hyperbolic group is quasi-isometrically embedded into the ambient Gromov hyperbolic space, in the sense that such event happens for eventual probability 1. See also [MS19] and [MT21] for additional conclusions under geometric assumptions, e.g., acylindricity or WPD. These results are linked to a deeper understanding of convex-cocompact subgroup of mapping class groups and outer automorphism groups, and random extensions of surface groups and free groups.

The following theorem strengthens the conclusion of Taylor-Tiozzo's theorem, while embracing more general spaces.

Theorem D. *Let (X, G, o) be as in Convention 1.1, and $\omega^{(1)}, \ldots, \omega^{(k)}$ be k independent random walk generated by a non-elementary measure μ on G. Then there exists $K > 0$ such that*

$$\mathbb{P}\left[\langle \omega_n^{(1)}, \ldots, \omega_n^{(k)} \rangle \text{ is q.i. embedded into a quasi-convex subset of } X\right] \geq 1 - Ke^{-n/K}.$$

Thanks to a concrete control of the decay rate, we can even deduce the analogous conclusion for counting problems.

Theorem E. *Let G be a finitely generated group acting on a metric space X with at least two independent contracting elements. Then for each $k > 0$, there exists a finite generating set S of G such that*

$$\frac{\#\left\{(g_1, \ldots, g_k) : \begin{array}{c} g_1, \ldots, g_k \in B_n(e), \langle g_1, \ldots, g_k \rangle \text{ is q.i. embedded} \\ \text{into a quasi-convex subset of } X \end{array}\right\}}{(\# B_n(e))^k}$$

converges to 1 exponentially fast.

For Gromov hyperbolic spaces and Teichmüller space, Theorem E for $k = 1$ has been observed in [Cho21b] as an affirmative answer to a version of Farb's conjecture in [Far06].

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2. Preliminaries

We continue to employ the notion of contracting isometries defined in [Cho22b];

Definition 2.1 (contracting sets)**.** *For a subset $A \subseteq X$ of a metric space X and $\epsilon > 0$, we define the* closest point projection *of $x \in X$ to A by*

$$\pi_A(x) := \{a \in A : d_X(x, a) = d_X(x, A)\}.$$

*A is said to be K-*contracting *if:*

(1) $\pi_A(z) \neq \emptyset$ for all $z \in X$ and

(2) for all $x, y \in X$ such that $d_X(x, y) \leq d_X(x, A) - K$ we have

$$\mathrm{diam}_X \left(\pi_A(x) \cup \pi_A(y)\right) \leq K.$$

*A K-contracting K-quasigeodesic is called a K-*contracting axis*. An isometry $g \in G$ is said to be K-*contracting *if its orbit $\{g^n o\}_{n\in\mathbb{Z}}$ is a K-contracting axis.*

We assume throughout the article that μ is a non-elementary probability measure on G, i.e., the support of μ generates a semigroup that contains two independent contracting isometries.

Definition 2.2 (Translation length)**.** *For $g \in G$, the* (asymptotic) translation length *of g is defined by*

$$\tau(g) := \liminf_{n\to\infty} \frac{1}{n} d(o, g^n o).$$

Given an isometry $g \in G$, the orbit $\{g^n o\}_{n\in\mathbb{Z}}$ is a quasi-geodesic if and only if g has strictly positive translation length. However, an isometry with positive translation length may not be contracting. For example, a translation in direction $(1, 1)$ on the Cayley graph of $\mathbb{Z}^2$ has translation length 2 but is not contracting.

Definition 2.3 (Quasi-convexity)**.** *A subset $A \subseteq X$ is said to be K-*quasi-convex *if any geodesic $[x, y]$ connecting two points $x, y \in A$ is contained in the K-neighborhood of A.*

Let us now recall some alignment lemmata established in [Cho22b].

Definition 2.4 ([Cho22b, Definition 3.2])**.** *Given paths κ_i from x_i to x_i' for each $i = 1, \ldots, n$, we say that $(\kappa_1, \ldots, \kappa_n)$ is C-aligned if*

$$\operatorname{diam}\left(x_i' \cup \pi_{\kappa_i}(\kappa_{i+1})\right) < C, \quad \operatorname{diam}\left(x_{i+1} \cup \pi_{\kappa_{i+1}}(\kappa_i)\right) < C.$$

hold for $i = 1, \ldots, n-1$.

Lemma 2.5 ([Cho22b, Lemma 3.3])**.** *For each $C > 0$ and $K > 1$, there exists $D = D(K, C) > C$ that satisfies the following property.*

Let κ, η be K-contracting axes that connect x to x' and y to y', respectively. Suppose that (κ, y') and (x, η) are C-aligned. Then (κ, η) is D-aligned.

Proposition 2.6 ([Cho22b, Proposition 3.5])**.** *For each $C > 0$ and $K > 1$, there exist $D = D(K, C) > C$ and $L = L(K, C) > C$ that satisfies the following.*

Let J be a nonempty set of consecutive integers, and $p, \{x_i, y_i\}_{i \in J}$ be points in X. For each $i \in J$, let κ_i be a K-contracting axis connecting x_i to y_i whose domain is longer than L. Suppose also that $(\kappa_i)_{i \in J}$ is C-aligned. Then we have the following:

(1) the statements

$$(\kappa_i, p) \text{ is } D\text{-aligned}, \quad (p, \kappa_i) \text{ is } D\text{-aligned}$$

cannot hold simultaneously;

(2) the set

$$\begin{aligned} J_0 &= J_0\Big(p; (\kappa_i)_{i \in J}, D\Big) \\ &:= \left\{ j \in J : \begin{array}{l} (\kappa_i, p) \text{ is } D\text{-aligned for } i \in J \text{ such that } i < j, \\ (p, \kappa_i) \text{ is } D\text{-aligned for } i \in J \text{ such that } i > j \end{array} \right\} \end{aligned}$$

consists of either a single integer or two consecutive integers;

(3) $\pi_{\cup_i \kappa_i}(p)$ is nonempty and is contained in $\bigcup\{\pi_{\kappa_j}(p) : j \in J_0\}$; and

(4) (κ_l, κ_m) is D-aligned for any $l, m \in J$ such that $l < m$.

Proposition 2.7 ([Cho22b, Proposition 3.6])**.** *For each $C > 0$ and $K > 1$, there exist $E = E(K, C) > C$ and $L = L(K, C) > C$ that satisfy the following. Let $x, y \in X$ and $\kappa_1, \ldots, \kappa_N$ be K-contracting axes whose domains are longer than L.*

If $(x, \kappa_1, \ldots, \kappa_N, y)$ is C-aligned, then (x, κ_i, y) is E-witnessed for each $i = 1, \ldots, N$. Moreover, $p \in \mathcal{N}_E([x, y])$ and $(x, y)_p < E$ for any $p \in \kappa_i$.

Lemma 2.8 ([Cho22b, Lemma 3.7])**.** *For each $C, M > 0$ and $K > 1$, there exist $K' = K'(K, C, M) > C$ and $L = L(K, C) > C$ that satisfies the following.*

Let J be a nonempty set of consecutive integers and $\{x_i, y_i\}_{i\in J}$ be points in X. For each $i \in J$, let κ_i be a K-contracting axis connecting x_i and y_i whose domain is longer than L. Suppose that $(\kappa_i)_{i\in J}$ is C-aligned and $d(y_i, x_{i+1}) < M$ for $i \in J \setminus \sup J$. Then $\cup_i \kappa_i$ is a K'-contracting axis.

We now recall the concept of Schottky sets. Given a sequence $s = (\phi_i)_{i=1}^k$ of isometries of X, we denote the product of its entries $\phi_1 \cdots \phi_k$ by $\Pi(s)$. We also define the reversal of s by $s^{-1} := (\phi_{k-i+1}^{-1})_{i=1}^k$, i.e.,

$$s = (\phi_1, \ldots, \phi_k) \Leftrightarrow s^{-1} = (\phi_k^{-1}, \ldots, \phi_1^{-1}).$$

Now let

$$x_{nk+i} := \Pi(s)^n \phi_1 \cdots \phi_i o = (\phi_1 \cdots \phi_k)^n \phi_1 \cdots \phi_i o$$

for each $n \in \mathbb{Z}$ and $i = 0, \ldots, k-1$. We let $\Gamma^m(s) := (x_0, x_1, \ldots, x_{mk})$ when $m \geq 0$ and $\Gamma^m(s) := (x_0, x_{-1}, \ldots, x_{mk})$ when $m < 0$. When $m = 1$, we usually omit the superscript and write $\Gamma(s) = (x_0, \ldots, x_k)$. Finally, let $\Gamma^{\pm\infty}(s) = (x_i)_{i\in\mathbb{Z}}$. Note that $\Gamma^{-m}(s) = \Gamma^m(s^{-1})$, and $\Gamma^m(s)$ is a concatenation of $|m|$ translates of $\Gamma(s)$ or its reverse.

Definition 2.9 ([Cho22b, Definition 3.11])**.** *Let $K > 0$ and $S \subseteq G^M$ be a set of sequences of M isometries. We say that S is K-*Schottky *if the following hold:*

(1) $\Gamma^m(s)$ is a K-contracting axis for all $s \in S$ and $m \in \mathbb{Z}$;
(2) for each $x \in X$, all element $s \in S$ except at most 1 satisfies that $(x, \Gamma^n(s))$ is K-aligned for all $n \in \mathbb{Z}$;
(3) for each $x \in X$ and $s \in S$, if $(x, \Gamma^n(s))$ is not K-aligned for some $n > 0$ ($n < 0$, resp.) then $(x, \Gamma^m(s))$ is K-aligned for all $m \leq 0$ ($m \geq 0$, resp.).

Proposition 2.10 ([Cho22b, Proposition 3.12])**.** *For any $N_0 > 0$, there exists a K-Schottky set of cardinality N_0 in $(\operatorname{supp}\mu)^m$ for some m and K.*

From now on we fix an integer $N_0 > 410$. Let $K_0 := K(N_0)$ be as in Proposition 2.10, and

- $D_0 := D(K_0, K_0)$ be as in Lemma 2.5;
- for $i = 1, 2$, $D_i := D(K_0, D_{i-1})$, $L_i := L(K_0, D_{i-1})$ be as in Lemma 2.5 and Proposition 2.6;
- $E_0 := E(K_0, D_2)$, $L_3 := L(K_0, D_2)$ be as in Proposition 2.7.

Let us now fix a K_0-Schottky set $S \subseteq (\operatorname{supp}\mu)^{M_0}$ of cardinality at least N_0.

Note that the set of n-self-concatenations of elements of S is also a K_0-Schottky set. Hence, we may assume that

$$M_0 > L_1 + L_2 + L_3 + 20K_0(K_0 + E_0). \tag{2.1}$$

From now on, K_0-contracting axes of the form $\Gamma^m(s)$ for $s \in S$ and $m \neq 0$ are called *Schottky axes.*

3. The first method: deviation inequalities

Our first approach to the limit laws for translation length does not explicitly refer to the pivotal times but implicitly rely on them via deviation inequalities.

We briefly recall the RV $\upsilon = \upsilon(\check{\omega}, \omega)$ defined in [Cho22b, Section 5]. Given independent backward and forward paths, it was defined as the minimal index k for which there exists $i \le k - M_0$ such that:

(1) $\alpha := (g_{i+1}, \ldots, g_{i+M_0})$ is a Schottky sequence;
(2) $(o, \omega_i \, \Gamma(\alpha), \omega_n \, o)$ is D_1-aligned for all $n \ge k$, and
(3) $(\check{\omega}_{n'} \, o, \omega_i \, \Gamma(\alpha))$ is D_2-aligned for all $n' \ge 0$.

Similarly, we defined $\check{\upsilon} = \check{\upsilon}(\check{\omega}, \omega)$ as the minimal index k that are associated with another index $i \le k$ such that:

(1) $\check{\alpha} := (\check{g}_{i+1}, \ldots, \check{g}_{i+M_0})$ is a Schottky sequence;
(2) $(o, \check{\omega}_i \Gamma(\check{\alpha}), \check{\omega}_n o)$ is D_1-aligned for all $n \ge k$, and
(3) $(\omega_n \, o, \check{\omega}_i \Gamma(\check{\alpha}))$ is D_2-aligned for all $n \ge 0$.

We then had the following results:

Lemma 3.1 ([Cho22b, Lemma 5.3])**.** *There exist $\kappa, K > 0$ such that the following estimate holds for all k:*

$$\mathbb{P}\left(\upsilon(\check{\omega}, \omega) \ge k \,\middle|\, g_{k+1}, \check{g}_1, \ldots, \check{g}_{k+1} \right) \le K e^{-\kappa k},$$
$$\mathbb{P}\left(\check{\upsilon}(\check{\omega}, \omega) \ge k \,\middle|\, \check{g}_{k+1}, g_1, \ldots, g_{k+1} \right) \le K e^{-\kappa k}.$$

Lemma 3.2 ([Cho22b, Corollary 5.6])**.** *Suppose that μ has finite p-moment for some $p > 0$. Then there exists $K > 0$ such that*

$$\mathbb{E}\left[\min\{ d(o, \omega_\upsilon \, o), d(o, \check{\omega}_{\check{\upsilon}} o) \}^{2p} \right] < K.$$

Using this, we can prove Theorem B.

Proof. Suppose first that μ has finite p-th moment for some $p > 0$. Let Z be an integrable RV that dominates $\min\{d(o, \omega_\upsilon \, o), d(o, \check{\omega}_{\check{\upsilon}} o)\}^{2p}$. Let also $\kappa_1, K_1 > 0$ be the constants as in Lemma 3.1 and 3.2.

Let us fix $n > 0$. We temporarily define

$$h_{nk+i} := g_i \quad (k \in \mathbb{Z}, i \in \{1, \ldots, n\}),$$
$$\omega_i := \begin{cases} h_1 \cdots h_i & i \ge 0, \\ h_0^{-1} \cdots h_{i+1}^{-1} & i < 0. \end{cases}$$

For $t = 0, 1, 2, 3$, we also define

$$\begin{array}{ll} g_{i;t} := h_{i + \lfloor nt/4 \rfloor} & \check{g}_{i;t} := h^{-1}_{\lfloor nt/4 \rfloor - i + 1} \\ \omega_{i;t} := g_{1;t} \cdots g_{i;t}, & \check{\omega}_{i;t} := \check{g}_{1;t} \cdots \check{g}_{i;t}. \end{array} \qquad (i = 1, \ldots, \lfloor n/2 \rfloor)$$

$(\check{g}_{i;t}, g_{i;t})_i$'s for $t = 0, 1, 2, 3$ have the same distribution with $(\check{g}_i, g_i)_i$, although they are not mutually independent. Let

$$\upsilon_{(t)} := \upsilon\left((\check{\omega}_{i;t})_{0 \le i \le \lfloor n/2 \rfloor}, (\omega_{i;t})_{0 \le i \le \lfloor n/2 \rfloor} \right), \quad \check{\upsilon}_{(t)} := \check{\upsilon}\left((\check{\omega}_{i;t})_{0 \le i \le \lfloor n/2 \rfloor}, (\omega_{i;t})_{0 \le i \le \lfloor n/2 \rfloor} \right)$$

and observe that

$$\mathbb{P}\left(A_{n;t} := \{\omega : \max\{\upsilon_{(t)}, \check{\upsilon}_{(t)} \geq n/10\}\right) \leq 2K_1 e^{-\kappa_1 n/10},$$
$$\min\left\{d\left(o, \omega_{\upsilon_{(0)}} o\right), d\left(o, \check{\omega}_{\check{\upsilon}_{(0)}} o\right)\right\}^{2p} \leq Z',$$

, where Z' is an RV of the same distribution with Z. We now claim that for $\omega \notin A_n^{(0)} \cup A_n^{(1)} \cup A_n^{(2)} \cup A_n^{(3)}$, we have

$$[d(o, \omega_n o) - \tau(\omega_n)]^{2p} \leq 2^{2p} Z'.$$

We explain the case that $d\left(o, \omega_{\upsilon_{(0)}} o\right)^{2p} \leq Z$, since the other case can be handled in a similar manner.

By the definition of $\upsilon_{(t)}$, there exist $i(0), i(1), i(2), i(3)$ such that $nt/4 \leq i(t) \leq nt/4 + \upsilon_{(t)} - M_0$ and the following holds. If we define

$$s_t = (g_{i(t)+1}, \ldots, g_{i(t)+M_0}),$$

then s_t's are Schottky sequences and

$$\left(\omega_{\lfloor nt/4 \rfloor - j}\, o,\ \omega_{i(t)}\, \Gamma(s_t),\ \omega_{\lfloor nt/4 \rfloor + k}\right)$$

is D_2-aligned for $0 \leq j \leq n/2$ and $\upsilon_{(t)} \leq k \leq n/2$. Note also that $\upsilon_{(t)} \leq n/10$ since w does not belong to any of $A_{n;t}$. This implies that

$$\left(o,\ \omega_{i(0)}\, \Gamma(s_0),\ \ldots,\ \omega_{i(3)}\, \Gamma(s_3),\ \omega_n\, \omega_{i(0)}\, \Gamma(s_1),\ \ldots,\ \omega_n^{k-1}\, \omega_{i(3)}\, \Gamma(s_3),\ \omega_n^k\, o\right)$$

is D_2-aligned for each $k > 0$. Using Proposition 2.7 we can control the Gromov products among points, which imply

$$d(o, \omega_n^k o) \geq d(o, \omega_{i(0)} o) + \sum_{j=1}^{k-1} d\left(\omega_n^{j-1} \omega_{i(0)} o, \omega_n^j \omega_{i(0)} o\right) + d(o, \omega_n^{k-1} \omega_{i(0)} o, \omega_n^k o) - (k+1) E_0.$$

Hence, we have

$$\tau(\omega_n) \geq d(\omega_{i(0)} o, \omega_n \omega_{i(0)} o) - E_0,$$
$$[d(o, \omega_n o) - \tau(\omega_n)]^{2p} \leq (2d(o, \omega_i o) + E_0)^{2p}.$$

Note that $(o, \omega_{i(0)} \Gamma(s_0), \omega_{\upsilon_{(0)}} o)$ is also D_2-aligned so we have

$$\begin{aligned} d(o, \omega_{i(0)} o) &= d(o, \omega_{\upsilon_{(0)}} o) - d(\omega_{i(0)} o, \omega_{i(0)+M_0} o) - d(\omega_{i(0)+M_0} o, \omega_{\upsilon_{(0)}} o) \\ &\quad + 2(o, \omega_{i(0)+M_0} o)_{\omega_{i(0)} o} + 2(o, \omega_{\upsilon_{(0)}} o)_{\omega_{i(0)+M_0} o} \\ &\leq d(o, \omega_{\upsilon_{(0)}} o) - 6E_0, \end{aligned}$$

$$[d(o, \omega_n o) - \tau(\omega_n)]^{2p} \leq (2d(o, \omega_i o) + E_0)^{2p} \leq 2^{2p} d(o, \omega_{\upsilon_{(0)}} o)^{2p} \leq 2^{2p} Z'.$$

Given this claim. we obtain

$$\begin{aligned} \mathbb{P}(d(o, \omega_n o) - \tau(\omega_n) \geq C n^{1/2p}) &= \mathbb{P}\left([d(o, \omega_n o) - \tau(\omega_n)]^{2p} \geq C^{2p} n\right) \\ &\leq \mathbb{P}\left(2^{2p} Z \geq C^{2p} n\right) + 2K_1 e^{-\kappa n/10}. \end{aligned}$$

Since Z is integrable, the above probability is summable and the Borel-Cantelli lemma leads to the conclusion.

Now suppose that μ has finite first moment. This time, we define

$$A_{n;t} := \{\omega : \upsilon_{(t)} \geq K' \log n\}$$

for some large K' such that $\sum_n K_1 e^{-\kappa_1 K' \log n} < +\infty$. Then the Borel-Cantelli lemma tells us that almost every path ω eventually lies outside $A_n^{(1)} \cup A_n^{(2)} \cup A_n^{(3)} \cup A_n^{(4)}$, say for $n \geq N$. In such case, we have

$$d(o, \omega_n o) - \tau(\omega_n) \leq d(o, \omega_{\upsilon_{(0)}} o) \leq d(o, \omega_{K' \log n} o)$$

for $n \geq N$. The subadditive ergodic theorem tells us that $d(o, \omega_m o) \leq 2\lambda m$ eventually holds for almost every path. Hence we conclude that

$$d(o, \omega_n o) - \tau(\omega_n) \leq 2\lambda K' \log n$$

eventually for almost every path. □

Corollary 3.3 (SLLN for finite first moment)**.** *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G with finite first moment. Then*

$$\lim_n \frac{1}{n} \tau(\omega_n) = \lambda \tag{3.1}$$

for almost every ω, where $\lambda = \lambda(\mu)$ is the escape rate of μ.

Corollary 3.4 (CLT)**.** *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. If μ has finite second moment, then $\frac{1}{\sqrt{n}}(\tau(o, \omega_n o) - n\lambda)$ and $\frac{1}{\sqrt{n}}(d(o, \omega_n o) - n\lambda)$ converge to the same Gaussian distribution $\mathscr{N}(0, \sigma(\mu)^2)$ in law. We also have*

$$\limsup_{n \to \infty} \pm \frac{\tau(o, \omega_n o) - \lambda n}{\sqrt{2n \log \log n}} = \sigma(\mu) \quad \textit{almost surely.}$$

In fact, Theorem B implies Corollary 3.3 for measures with finite (1/2)-th moment, and the converse of CLT for measures with finite (1/4)-th moment. For general non-elementary measures, however, the SLLN and the converse of CLT cannot be deduced from Theorem B and we need more explicit information. Let us recall the following result from [Cho22b]:

Theorem F ([Cho22b, Theorem E])**.** *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. Then for any $0 < L < \lambda(\omega)$, there exists $K > 0$ such that*

$$\mathbb{P}[d(o, \omega_n o) \leq Ln] \leq K e^{-n/K}$$

holds.

Using this theorem, let us prove Theorem A.

Proof. Given $0 < L < \lambda(\omega)$, we fix $0 < \epsilon < 1/10$ such that $L' = L/(1 - 2\epsilon)$ is still smaller than λ.

Let us define $A_{n;t} := \{\omega : \max\{\upsilon_{(t)}, \check{\upsilon}_{(t)}\} \geq \epsilon n\}$. Then $\mathbb{P}(A_{n;t})$ decays exponentially. Now for $\omega \notin \cup_{t=1}^{4} A_{n;t}$, we have $i(t), \check{i}(t)$ such that

$$n/4 - \check{\upsilon}_{(t)} \leq \check{i}(t) \leq n/4 - M_0 \leq nt/4 \leq i(t) \leq nt/4 + \upsilon_{(t)} - M_0,$$

and the following holds. If we define

$$s_t = (h_{i(t)+1}, \ldots, h_{i(t)+M_0}),\ \check{s}_t = (h_{\check{i}(t)+1}, \ldots, h_{\check{i}(t)+M_0})$$

then s_t, $\check{s}_t$'s are Schottky sequences and

$$\Big(\omega_{\lfloor nt/4\rfloor - j}\, o,\ \omega_{\check{i}(t)}\, \Gamma(\check{s}_t),\ \omega_{i(t)}\, \Gamma(s_t),\ \omega_{\lfloor nt/4\rfloor + k}\Big)$$

is D_2-aligned for $\check{\upsilon}_{(t)} \leq j \leq n/2$ and $\upsilon_{(t)} \leq k \leq n/2$. This implies that

$$\left(\begin{array}{c} o,\ \omega_{i(0)}\, \Gamma(s_0),\ \ldots,\ \omega_{i(3)}\, \Gamma(s_3),\ \omega_{n-\check{i}(0)}\, \Gamma(\check{s}_0),\ \omega_n\, \omega_{i(0)}\, \Gamma(s_1), \\ \ldots,\ \omega_n^{k-1}\, \omega_{i(3)}\, \Gamma(s_3),\ \omega_n^{k-1}\, \omega_{n-\check{i}(0)}\, \Gamma(\check{s}_0),\ \omega_n^k\, o \end{array}\right)$$

is D_2-aligned for each $k > 0$. This implies that

$$\begin{aligned} \tau(\omega_n) &\geq d(\omega_{i(0)}\, o, \omega_{n-\check{i}(0)}\, o) - E_0 \\ &\geq \min\{d(\omega_i\, o, \omega_{n-j}\, o) : 0 \leq i, j \leq \epsilon n\} - E_0. \end{aligned} \tag{3.2}$$

Now, since the displacement satisfies Theorem F, there exists $K > 0$ such that

$$\mathbb{P}[d(o, \omega_m\, o) \leq L' m] \leq K e^{-m/K}$$

holds for all m. This implies

$$\begin{aligned} &\mathbb{P}[\min\{d(\omega_i\, o, \omega_{n-j}\, o) : 0 \leq i, j \leq \epsilon n\} - E_0 \leq Ln] \\ &\leq \sum_{0 \leq i,j \leq \epsilon n} \mathbb{P}[d(\omega_i\, o, \omega_{n-j}\, o) \leq Ln + E_0] \\ &\leq (\epsilon n)^2 \cdot K e^{-(1-2\epsilon)n/K} \end{aligned} \tag{3.3}$$

for large enough n, which decays exponentially. By combining Inequality 3.2, 3.3 and the exponential decay of $\mathbb{P}(A_{n;t})$, we deduce that $\mathbb{P}[\tau(\omega_n) \leq Ln]$ decays exponentially.

Meanwhile, we observed that if $\omega \notin \cup_{t=1}^{4} A_{n;t}$, then

$$(\ldots,\quad \omega_n^{k-1}\, \omega_{i(0)}\, \Gamma(s_0),\quad \omega_n^k\, \omega_{i(0)}\, \Gamma(s_0),\quad \ldots)$$

is a subsequence of a D_2-aligned sequence of Schottky axes. Since $d(\omega_n^{k-1}\, \omega_{i(0)}\, \Gamma(s_0), \omega_n^k\, \omega_{i(0)})$ is uniformly bounded, Proposition 2.6 tells us that ω_n is contracting. □

The previous proof did not rely on the possibility that the initial or the final segment of a random path is shorter than the middle one; indeed, if the random walk has no moment condition one cannot hope that. Instead, the proof explicitly used the fact that the middle segment will catch up the escape rate regardless of the moment condition, which is proven using the pivoting technique.

In order to discuss the converse of CLT for general measures, one should perform the pivoting more explicitly. For this purpose, we will recall the basics of the pivotal time construction in [Cho22b].

4. The second method: pivoting technique

4.1. **Pivotal times and pivoting.** This subsection is a summary of results in [Cho22b, Subsection 4.1]; for complete proofs, refer to the explanation there.

Let $(w_i)_{i=0}^{\infty}$, $(v_i)_{i=1}^{\infty}$ be isometries in G; these will be fixed throughout this subsection. Now given a sequence

$$s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_n, \beta_n, \gamma_n, \delta_n) \in S^{4n},$$

we first define

$$a_i := \Pi(\alpha_i),\ b_i := \Pi(\beta_i)\ c_i := \Pi(\gamma_i),\ d_i := \Pi(\delta_i). \tag{4.1}$$

We then consider isometries that are subwords of

$$w_0 a_1 b_1 v_1 c_1 d_1 w_1 \cdots a_k b_k v_k c_k d_k w_k \cdots .$$

More precisely, we set the initial case $w_{-1,2}^{+} := id$ and define

$$\begin{aligned} w_{i,2}^{-} &:= w_{i-1,2}^{+} w_{i-1}, & w_{i,1}^{-} &:= w_{i,2}^{-} a_i, & w_{i,0}^{-} &:= w_{i,2}^{-} a_i b_i, \\ w_{i,0}^{+} &:= w_{i,2}^{-} a_i b_i v_i, & w_{i,1}^{+} &:= w_{i,2}^{-} a_i b_i v_i c_i, & w_{i,2}^{+} &:= w_{i,2}^{-} a_i b_i v_i c_i d_i. \end{aligned}$$

We also employ notations

$$\begin{aligned} \Upsilon(\alpha_i) &:= w_{i,2}^{-}\Gamma(\alpha_i), & \Upsilon(\beta_i) &:= w_{i,1}^{-}\Gamma(\beta_i), \\ \Upsilon(\gamma_i) &:= w_{i,0}^{+}\Gamma(\gamma_i), & \Upsilon(\delta_i) &:= w_{i,1}^{+}\Gamma(\delta_i). \end{aligned}$$

for simplicity.

We then defined the set $P_n = P_n(s, (w_i)_i, (v_i)_i) \subseteq \{1, \ldots, n\}$. Our main estimates were as follows.

Lemma 4.1 ([Cho22b, Lemma 4.1])**.** *Let $P_n = \{i(1) < \ldots < i(m)\}$. Then*

$$\Big(o, \Upsilon(\alpha_{i(1)}), \Upsilon(\beta_{i(1)}), \Upsilon(\gamma_{i(1)}), \Upsilon(\delta_{i(1)}), \ldots, \Upsilon(\alpha_{i(m)}), \Upsilon(\beta_{i(m)}), \Upsilon(\gamma_{i(m)}), \Upsilon(\delta_{i(m)}), y_{n+1,2}^{-}\Big)$$

is a subsequence of a D_0-aligned sequence of Schottky axes. In particular, it is D_1-aligned.

In [Cho22b], we have observed a sufficient condition for $P_k = P_{k-1} \cup \{k\}$ to hold. Namely, the conditions

$$\operatorname{diam}\Big(\pi_{\Upsilon(\gamma_k)}(y_{k,0}^{-}) \cup y_{k,0}^{+}\Big) = \operatorname{diam}\Big(\pi_{\Gamma(\gamma_k)}(v_k^{-1}o) \cup o\Big) < K_0, \tag{4.2}$$

$$\operatorname{diam}\Big(\pi_{\Upsilon(\delta_k)}(y_{k+1,2}^{-}) \cup y_{k,2}^{+}\Big) = \operatorname{diam}\Big(\pi_{\Gamma^{-1}(\delta_k)}(w_k o) \cup o\Big) < K_0, \tag{4.3}$$

$$\operatorname{diam}\Big(\pi_{\Upsilon(\beta_k)}(y_{k,1}^{+}) \cup y_{k,0}^{-}\Big) = \operatorname{diam}\Big(\pi_{\Gamma^{-1}(\beta_k)}(v_k c_k o) \cup o\Big) < K_0, \tag{4.4}$$

$$\operatorname{diam}\Big(\pi_{\Upsilon(\alpha_k)}(z_{k-1}) \cup y_{k,2}^{-}\Big) = \operatorname{diam}\Big(\pi_{\Gamma(\alpha_k)}\Big((w_{k,2}^{-})^{-1} z_{k-1}\Big) \cup o\Big) < K_0. \tag{4.5}$$

guaranteed the addition of k to the set of pivotal times. Each condition excluded at most one element out of the Schottky set S and we obtained:

Lemma 4.2 ([Cho22b, Lemma 4.2]). *For $1 \le k \le n$, $s \in S^{4(k-1)}$, we have*

$$\mathbb{P}\left(\#P_k(s, \alpha_k, \beta_k, \gamma_k, \delta_k) = \#P_{k-1}(s) + 1\right) \ge 1 - 4/N_0.$$

Given $\alpha_1,\ \beta_1,\ \gamma_1,\ \delta_1,\ \ldots,\ \alpha_{k-1},\ \beta_{k-1},\ \gamma_{k-1}, \delta_{k-1}$, we define the set $\tilde{S}_k$ of triples $(\alpha_k, \beta_k, \gamma_k)$ in S^3 that satisfy Condition 4.2, 4.4 and 4.5. We then observed that $\#\left[S^3 \setminus \tilde{S}_k\right] \le 3(\#S)^2$. Moreover, for $(\alpha_k, \beta_k, \gamma_k) \in \tilde{S}_k$, $\{(\alpha_k, \beta'_k, \gamma_k) \in \tilde{S}_k : \beta_k \in S\}$ has at least $\#S - 1$ elements. We finally had:

Lemma 4.3 ([Cho22b, Lemma 4.3]). *Let $i \in P_k(s)$ for a choice $s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_n, \beta_n, \gamma_n, \delta_n)$, and $\bar{s}$ be obtained from s by replacing $(\alpha_i, \beta_i, \gamma_i)$ with*

$$(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i) \in \tilde{S}_i(\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}, \delta_{i-1}).$$

Then $P_l(s) = P_l(\bar{s})$ and $\tilde{S}_l(s) = \tilde{S}_l(\bar{s})$ for each $1 \le l \le k$.

Given $1 \le k \le n$ and a partial choice $s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_k, \beta_k, \gamma_k, \delta_k)$, we defined pivoting as follows: $\bar{s} = (\bar{\alpha}_1, \bar{\beta}_1, \bar{\gamma}_1, \bar{\delta}_1, \ldots, \bar{\alpha}_k, \bar{\beta}_k, \bar{\gamma}_k, \bar{\delta}_k)$ is *pivoted from s* if:

- $\delta_j = \bar{\delta}_j$ for all $1 \le j \le k$,
- $(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i) \in \tilde{S}_i(s)$ for each $i \in P_k(s)$, and
- $(\alpha_j, \beta_j, \gamma_j) = (\bar{\alpha}_j, \bar{\beta}_j, \bar{\gamma}_j)$ for each $j \in \{1, \ldots, k\} \setminus P_k(s)$.

Lemma 4.3 then asserted that being pivoted from each other is an equivalence relation.

Corollary 4.4. *When $s = (\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^n$ is chosen from S^{4n} with the uniform measure, $\#P_n(s)$ is greater in distribution than the sum of n i.i.d. X_i, whose distribution is given by*

$$\mathbb{P}(X_i = j) = \begin{cases} (N_0 - 4)/N_0 & \text{if } j = 1, \\ (N_0 - 4)4^{-j}/N_0^{-j+1} & \text{if } j < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

More generally, the distribution of $\#P_{k+n}(s) - \#P_k(s)$ conditioned on the choices of $(\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^k$ also dominates the sum of n i.i.d. X_i.

Moreover, we have $\mathbb{P}(\#P_n(s) \le (1 - 10/N_0)n) \le e^{-Kn}$ for some $K > 0$.

4.2. **Pivoting and self-repulsion.** We discuss pivoting on random paths for translation length. Given $(w_j)_{j=0}^\infty$, $(v_j)_{j=0}^\infty$, we consider an equivalence class $\mathcal{E} \subseteq S^{4n}$ made by pivoting. $\mathcal{E}$ has a well-defined set of pivotal times $P_n(\mathcal{E}) = \{i(1), \ldots, i(M)\}$, and a choice $s \in \mathcal{E}$ is determined by the choices $(\alpha_{i(l)}, \beta_{i(l)}, \gamma_{i(l)})_{l=1}^M$. We also denote $w_{n+1,2}^-(s)$ by w for convenience throughout the subsection.

Recall that we have constructed $\tilde{S}_{i(l)} \subseteq S^3$ that depends on $\big(\alpha_{i(j)}, \beta_{i(j)}, \gamma_{i(j)}\big)_{j=1}^{l-1}$. We now define new subsets:

$$\begin{aligned}
S_1^*(s) &= S_1^*(\gamma_{i(M)}),\\
S_M^*(s) &= S_M^*(\alpha_{i(1)}, \gamma_{i(M)}),\\
S_2^*(s) &= S_2^*(\alpha_{i(1)}, \beta_{i(1)}, \gamma_{i(1)}, \alpha_{i(M)}, \beta_{i(M)}, \gamma_{i(M)}, \gamma_{i(M-1)}),\\
S_{M-1}^*(s) &= S_{M-1}^*(\alpha_{i(1)}, \beta_{i(1)}, \gamma_{i(1)}, \alpha_{i(M)}, \beta_{i(M)}, \gamma_{i(M)}, \alpha_{i(2)}, \gamma_{i(M-1)}),\\
&\ \ \vdots
\end{aligned}$$

for $1 \le k \le \lfloor M/2 \rfloor$. To define them we first consider

$$\begin{aligned}
\phi_k &:= (w^-_{i(M-k+1),0})^{-1} w w^-_{i(k),2}\\
&= v_{i(M-k+1)} c_{i(M-k+1)} d_{i(M-k+1)} w_{i(M-k+1)} \cdots a_n b_n v_n c_n d_n w_n\\
&\quad \cdot w_0 a_1 b_1 v_1 c_1 d_1 w_1 \cdots a_{i(k)-1} b_{i(k)-1} v_{i(k)-1} c_{i(k)-1} d_{i(k)-1} w_{i(k)-1}
\end{aligned}$$

for $1 \le k \le \lfloor M/2 \rfloor$. It is clear that ϕ_k depends on $\gamma_{i(M-k+1)}$, $\alpha_{i(M-k+2)}$, $\ldots$, $\gamma_{i(M)}$, $\alpha_{i(1)}$, $\beta_{i(1)}$, $\ldots$, $\gamma_{i(k-1)}$. Then we set

$$\begin{aligned}
S_k^*(s) &:= \Big\{\alpha_{i(k)} \in S \quad : \Big(w^{-1} y^-_{i(M-k+1),0},\ \Upsilon(\alpha_{i(k)})\Big) \text{ is } K_0\text{-aligned}\Big\},\\
S_{M-k+1}^*(s) &:= \Big\{\beta_{i(M-k+1)} \in S : \Big(w^{-1}\,\Upsilon(\beta_{i(M-k+1)}), y^-_{i(k),1}\Big) \text{ is } K_0\text{-aligned}\Big\}.
\end{aligned}$$

Here, the conditions above can be expressed as

$$\operatorname{diam}\Big(\pi_{\Gamma(\alpha_{i(k)})}(\phi_k^{-1} o) \cup o\Big) < K_0, \tag{4.7}$$

$$\operatorname{diam}\Big(\pi_{\Gamma^{-1}(\beta_{i(M-k+1)})}(\phi_k a_{i(k)} o) \cup o\Big) < K_0, \tag{4.8}$$

respectively. For each l, $S \setminus S_l^*(s)$ consists of at most 1 element thanks to the property of the Schottky set S.

Lemma 4.5. *Let $1 \le k \le M/2$. Suppose that $s = (\alpha_{i(l)}, \beta_{i(l)}, \gamma_{i(l)})_{l=1}^M \in \mathcal{E}_n$ satisfies*

$$\alpha_{i(k)} \in S_k^*(s), \quad \beta_{i(M-k+1)} \in S_{M-k+1}^*(s).$$

Then $w = w^-_{n+1,2}$ is a contracting isometry and satisfies

$$\tau(w) \ge d(o, wo) - \Big[d\big(o, y^-_{i(k),1}\big) + d\big(y^-_{i(M-k+1),1}, wo\big)\Big] - 4E_0.$$

Proof. Suppose that $s \in \mathcal{E}_n$ satisfies the hypothesis. Then by Lemma 2.5, $\big(w^{-1}\,\Upsilon(\beta_{i(M-k+1)}), \Upsilon(\alpha_{i(k)})\big)$ is D_0-aligned. Recall also that

$$\big(\Upsilon(\alpha_{i(k)}), \Upsilon(\beta_{i(k)}), \Upsilon(\gamma_{i(k)}), \Upsilon(\delta_{i(k)}), \ldots, \Upsilon(\alpha_{i(M-k+1)}), \Upsilon(\beta_{i(M-k+1)}), \Upsilon(\gamma_{i(M-k+1)}), \Upsilon(\delta_{i(M-k+1)})\big)$$

is a subsequence of a D_1-aligned sequence by Lemma 4.1. Hence, if we define

$$\begin{aligned}
\kappa_{2t+1} &:= w^t\,\Upsilon(\alpha_{i(k)}),\\
\kappa_{2t+2} &:= w^t\,\Upsilon(\beta_{i(M-k+1)})
\end{aligned}$$

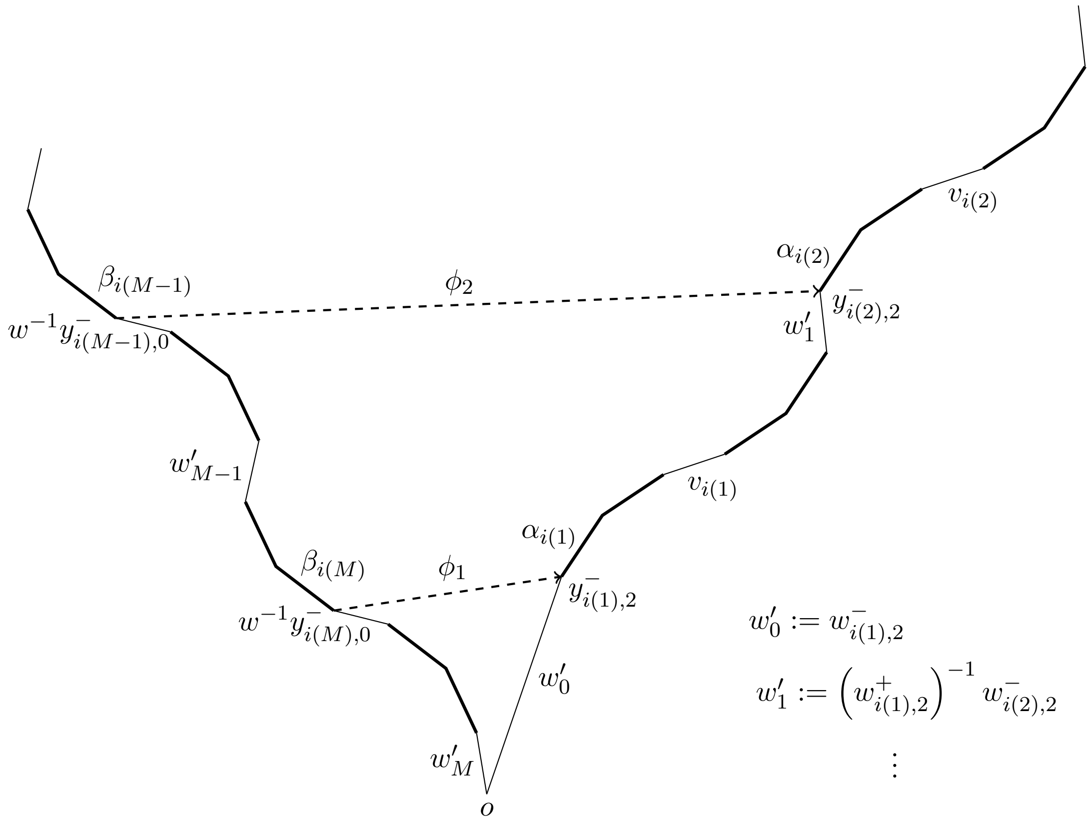

FIGURE 1. Defining ϕ_k's used in the pivoting for translation length.

for $t \in \mathbb{Z}$, we observe that $(o, \kappa_1, \kappa_2, \ldots, \kappa_{2i-1}, \omega^i o)$ is a subsequence of a D_1-aligned sequence. Proposition 2.7 then tells us that the Gromov products among the endpoints of κ_i's are bounded by E_0. Hence, we have

$$d(o, w^i o) \geq d(o, y^-_{i(k),1}) + \sum_{j=1}^{i} d(w^{j-1} y^-_{i(k),1}, w^{j-1} y^-_{i(M-k+1),1}) + \sum_{j=1}^{i-1} d(w^{j-1} y^-_{i(M-k+1),1}, w^j y^-_{i(k),1}) + d(w^{i-1} y^-_{i(M-k+1),1}, w^i o) - 4iE_0.$$

Dividing the both hand sides by i, we conclude that

$$\begin{aligned} \tau(w) &\geq d(y^-_{i(k),1}, y^-_{i(M-k+1),1}) - 4E_0 \\ &\geq d(o, wo) - d(o, y^-_{i(k),1}) - d(y^-_{i(M-k+1),1}, o) - 4E_0. \end{aligned} \tag{4.9}$$

Moreover, since $[y^-_{i(k),1}, y^-_{i(M-k+1),1}]$ is E_0-witnessed by Schottky axes and longer than $4E_0$, Inequality 4.9 also tells us that $\tau(w) > 0$. Similarly we have $\tau(w^{-1}) > 0$, so w is a bi-quasigeodesic.

Now Lemma 2.8 tells us that the concatenation of κ_i's is a contracting axis. This implies that ω_n is a contracting element. □

We now estimate the probability for the event described in Lemma 4.5. Given a choice

$$\bar{s} = (\bar{\alpha}_{i(l)}, \bar{\beta}_{i(l)}, \bar{\gamma}_{i(l)})_{l=1,\ldots,k-1,M-k+2\ldots,M} \in \tilde{S}_{i(1)} \times \cdots \times \tilde{S}_{i(k-1)} \times \tilde{S}_{i(M-k+2)} \times \cdots \times \tilde{S}_{i(M)},$$

we define

$$S_k^\dagger := \left\{ \begin{array}{c} (\alpha_{i(k)}, \beta_{i(k)}, \gamma_{i(k)}, \alpha_{i(M-k+1)}, \beta_{i(M-k+1)}, \gamma_{i(M-k+1)}) \in \tilde{S}_{i(k)} \times \tilde{S}_{i(M-k+1)} \\ : \quad \alpha_{i(k)} \in S_k^*(\bar{s}, \gamma_{M-k+1}) \text{ and} \\ \beta_{i(M-k+1)} \in S_{M-k+1}^*(\bar{s}, \alpha_{i(k)}, \gamma_{i(M-k+1)}) \end{array} \right\}$$

Then we have the following:

Lemma 4.6. *For each $1 \le k \le \lfloor M/2 \rfloor$, the cardinality of $\tilde{S}_k^\dagger$ is at least $(\#S)^6 - 8(\#S)^5$.*

Proof. First, there are at least $(\#S-1)$ choices of $\gamma_{i(k)}$ and $(\#S-1)$ choices of $\gamma_{i(M-k+1)}$ in S that satisfy Inequality 4.2. Fixing those choices, at least $(\#S-1)$ choices of $\beta_{i(k)}$ in S satisfy Inequality 4.4. Finally, fixing those choices, there are at most 1 choice of $\alpha_{i(k)}$ in S that violates Inequality 4.5 and at most 1 choice that violates Inequality 4.7. In other words, at least $(\#S-2)$ choices of $\alpha_{i(k)}$ satisfy both inequalities.

Fixing the above choices, at most 1 choices of $\beta_{i(M-k+1)}$ in S violates Inequality 4.4 and at most 1 choice in S violates Inequality 4.8. In other words, at least $(\#S-2)$ choices of $\beta_{i(k)}$ satisfy both inequalities. Finally, fixing those choices, there are at least $(\#S-1)$ choices of $\alpha_{i(k)}$ in S that satisfy Inequality 4.5. Overall, we conclude that $\tilde{S}_k^\dagger$ has cardinality at least $(\#S-1)^4(\#S-2)^2 \ge (\#S)^6 - 8(\#S)^5$. □

4.3. **A variation: v-pivoting.** We now fix subsets $S_1, S_2 \subseteq S$ of cardinality at least $N_0/4$, and a subset $A \subseteq G$. We then assume that for each $s_1 \in S_1, s_2 \in S_2$ and $v \in A$, the two sequences

$$(v^{-1}o, \Gamma(s_2)), \quad (v\Pi(s_2)o, \Gamma^{-1}(s_1)) \tag{4.10}$$

are K_0-aligned.

As in Subsection 4.1, we consider the subwords of

$$w_0 a_1 b_1 v_1 c_1 d_1 \cdots a_n b_n v_n c_n d_n w_n \cdots$$

and define $w_{i,j}^\pm$, $y_{i,j}^\pm$ analogously. This time, however, w_i's are chosen from G and *v_i's are chosen from A*. Also, we will not fix the choice of $(v_i)_i$ this time; only $(w_i)_i$ is fixed. Also, α_i, β_i's are chosen from S_1 and γ_i, δ_i's are chosen from S_2. In other words, a choice $s = (\alpha_1, \beta_1, \ldots, \gamma_n, \delta_n)$ is drawn from $(S_1^2 \times S_2^2)^n$.

Given a choice s, we construct the set of pivotal times $P_n = P_n(s, (w_i)_i, (v_i)_i)$ (with an auxiliary moving point z_n) as in Subsection 4.1. Then all the lemmata are intact except for some probabilistic estimates. For example, in

Lemma 4.2 we now have

$$\mathbb{P}\left(\#P_k(s,\alpha_k,\beta_k,\gamma_k,\delta_k) = \#P_{k-1}(s)+1\right) \ge 1 - 16/N_0,$$

since the choices $\alpha_k, \beta_k, \gamma_k, \delta_k$ are drawn from S_1 or S_2, not the entire S. This also affects Corollary 4.4 accordingly. Meanwhile, we have the following variant of Lemma 4.3:

Lemma 4.7. *Let $i \in P_k(s, \mathbf{v})$ for a choice $s = (\alpha_1, \ldots, \delta_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$. If $\mathbf{v}' = (v_1', \ldots, v_n')$ is made from $\mathbf{v}$ by replacing v_i with an element of A, then $P_l(s,\mathbf{v}) = P_l(s,\mathbf{v}')$ and $\tilde{S}_l(s,\mathbf{v}) = \tilde{S}_l(s,\mathbf{v}')$ for each $1 \le l \le k$.*

Proof. Since $v_1, \ldots, v_{i-1}$ are intact, $P_l(s) = P_l(\bar{s})$ and $\tilde{S}_l'(s,\mathbf{v}) = \tilde{S}_l'(s,\mathbf{v}')$ hold for $l = 0, \ldots, i-1$. At step i, δ_i satisfies Condition 4.3 and $\bar{\alpha}_i$ satisfies 4.5 since $i \in P_k(s,\mathbf{v})$. Moreover, β_i and γ_i still satisfy Condition 4.2 and 4.4 after changing v_i into any other element in A, since we assumed Condition 4.10. Hence, i is newly added in $P_i(s,\mathbf{v}')$ and

$$P_i(s,\mathbf{v}') = P_{i-1}(s,\mathbf{v}') \cup \{i\} = P_{i-1}(s,\mathbf{v}) \cup \{i\} = P_i(s,\mathbf{v}').$$

We also have $\tilde{S}_i(s) = \tilde{S}_i(\bar{s})$ as z_{i-1}, $w_{i,2}^-$ are not affected, and Condition 4.2, 4.4 holds for all $\beta_i \in S_1$ and $\gamma_i \in S_2$ thanks to Condition 4.10.

Meanwhile, z_i is modified into $\bar{z}_i = \bar{y}_{i,1}^+ = g y_{i,1}^+ = g z_i$, where $g := w_{i,2}^- a_i b_i v_i' (w_{i,2}^- a_i b_i v_i)^{-1}$. More generally, we have

$$\begin{aligned} \bar{w}_{l,t}^- &= g w_{l,t}^- \quad (t \in \{0,1,2\},\, l > i), \\ \bar{w}_{l,0}^+ &= g w_{l,0}^+ \qquad\qquad\quad (l > i), \\ \bar{w}_{l,t}^+ &= g w_{l,t}^+ \qquad (t \in \{1,2\}, l \ge i). \end{aligned} \tag{4.11}$$

Now the rest of the proof of Lemma 4.3 in [Cho22b] applies here. ◻

Given a choice $s = (\alpha_1, \ldots, \delta_n) \in (S_1^2 \times S_2^2)^n$ and $\mathbf{v} = (v_i)_{i=1}^n \in A^n$, we say that $(s, \mathbf{v}')$ is *v-pivoted from* $(s,\mathbf{v})$ if $\mathbf{v}'$ differs from $\mathbf{v}$ only at the pivotal times for $(s,\mathbf{v})$. Then Lemma 4.7 tells us that being v-pivoted from each other is an equivalence relation that preserves the set of pivotal times.

4.4. **Converse of CLT.**

We are now ready to present the converse of CLT for displacement and translation length.

Proposition 4.8. *Let ω be the random walk on G generated by a non-elementary measure μ with infinite second moment. Then for any sequence $(c_n)_n$ of real numbers, both $\frac{1}{\sqrt{n}}(d(o,\omega_n o) - c_n)$ and $\frac{1}{\sqrt{n}}(\tau(\omega_n) - c_n)$ do not converge in law.*

Proof. For each pair of subsets S_1, S_2 of S with cardinality $N_0/2$, we define

$$A(S_1,S_2) := \left\{ g \in G : (\Gamma(s_1), \Pi(s_1) g\, \Pi(s_2) o) \text{ and } (g^{-1}o, \Gamma(s_2)) \text{ are } K_0\text{-aligned for } s \in S' \right\}.$$

Given an element g of G, there exist at least $N_0 - 1$ Schottky choices $s_2 \in S$ that makes $(g^{-1}o, \Gamma(s_2))$ K_0-aligned. Choosing $N_0/2$ choices $s_2^{(1)}, \ldots, s_2^{(N_0/2)}$

among them, we now want $(g\,\Pi(s_2^{(i)})o, \Gamma^{-1}(s_1))$ to be K_0-aligned for each $i = 1, \ldots, N_0/2$: there exist at least $N_0/2$ Schottky choices realizing them. As a result, each $g \in G$ belongs to $A(S_1, S_2)$ for some subsets $S_1, S_2 \in \binom{S}{N_0/2}$. Hence, we have

$$\sum_{\substack{S_1, S_2 \subseteq S \\ \#S' = N_0/2}} \sum_{g \in A(S_1, S_2)} \mu(g) d(o, go)^2 \geq \sum_{g \in G} \mu(g) d(o, go)^2 = +\infty,$$

which implies that

$$\mathbb{E}\left[d(o, go)^2 \,|\, g \in A(S_1, S_2)\right] = +\infty$$

for some $S_1, S_2 \subseteq S$ with cardinality $N_0/2$. Let μ_{S_1} and μ_{S_2} be the uniform measure on S_1 and S_2, respectively, and

$$\mu' := \begin{cases} \mu(g)/\mu(A(S_1, S_2)) & g \in A(S_1, S_2) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbb{E}_{\mu'}[d(o, go)^2] = +\infty$ and $\mu' \leq \frac{1}{\mu(A(S_1, S_2))}\mu$ hold. We now consider the decomposition

$$\mu^{(4M_0+1)} = \alpha(\mu_{S_1}^2 \times \mu' \times \mu_{S_2}^2) + (1 - \alpha)\nu$$

for some $0 < \alpha < 1$ and ν. We then consider:

- Bernoulli RVs ρ_i with $\mathbb{P}(\rho_i = 1) = \alpha$ and $\mathbb{P}(\rho_i = 0) = 1 - \alpha$,
- η_i with the law $\mu_{S_1}^2 \times \mu' \times \mu_{S_2}^2$, and
- ν_i with the law ν,

all independent, and define

$$(g_{(4M_0+1)k+1}, \ldots, g_{(4M_0+1)(k+1)}) = \begin{cases} \nu_k & \text{when } \rho_k = 0, \\ \eta_k & \text{when } \rho_k = 1. \end{cases}$$

Then $(g_i)_{i=1}^{\infty}$ has the law μ^{∞}. We now define Ω to be the ambient probability space on which the above RVs are all measurable. We will denote an element of Ω by ω. We also fix

- $\omega_k := g_1 \cdots g_k$,
- $\mathscr{B}(k) := \sum_{i=0}^{k} \rho_i$, i.e., the number of the Schottky slots till k, and
- $\vartheta(i) := \min\{j \geq 0 : \mathscr{B}(j) = i\}$, i.e., the i-th Schottky slot.

For each $\omega \in \Omega$ and $i \geq 1$ we define

$$\begin{aligned} w_{i-1} &:= g_{4M_0[\vartheta(i-1)+1]+1} \cdots g_{4M_0\,\vartheta(i)}, \\ \alpha_i &:= (g_{4M_0\,\vartheta(i)+1}, \ldots, g_{4M_0\,\vartheta(i)+M_0}), \\ \beta_i &:= (g_{4M_0\,\vartheta(i)+M_0+1}, \ldots, g_{4M_0\,\vartheta(i)+2M_0}), \\ v_i &:= g_{4M_0\,\vartheta(i)+2M_0+1}, \\ \gamma_i &:= (g_{4M_0\,\vartheta(i)+2M_0+2}, \ldots, g_{4M_0\,\vartheta(i)+3M_0+1}), \\ \delta_i &:= (g_{4M_0\,\vartheta(i)+3M_0+2}, \ldots, g_{4M_0\,\vartheta(i)+4M_0+1}). \end{aligned}$$

In other words, $\eta_{\vartheta(i)}$ corresponds to $(\alpha_i, \beta_i, v_i, \gamma_i, \delta_i)$ and w_i corresponds to the products of intermediate steps ν_k's in between $\eta_{\vartheta(i-1)}$ and $\eta_{\vartheta(i)}$. As in Section 4.1, we employ the notation $a_i := \Pi(\alpha_i)$, $b_i := \Pi(\delta_i)$ and so on.

In order to represent ω_n for arbitrary n, we set $n' := \lfloor n/4M_0 \rfloor - 1$ and $w^{(n)} := g_{4M_0[\vartheta(\mathscr{B}(n'))+1]+1} \cdots g_n$. We then have

$$\omega_n = w_0 a_1 b_1 v_1 c_1 d_1 w_1 \cdots a_{\mathscr{B}(n')} b_{\mathscr{B}(n')} c_{\mathscr{B}(n')} d_{\mathscr{B}(n')} w^{(n)} \tag{4.12}$$

and we can bring the discussion in Subsection 4.3 here. Explicitly, we first fix the choices of ρ_i's and ν_i's; this determines $\mathscr{B}(n')$ and the isometries $(w_0, \ldots, w^{(n)}), (v_1, \ldots, v_n)$. Then we consider the set of pivotal times $P_{\mathscr{B}(n')}(s)$ for $s \in (S_1^{(2)} \times S_2^{(2)})^n$. After this process, we define

$$\mathcal{P}_n(\omega) := \left\{ (4M_0 + 1)\, \vartheta(i) : i \in P_{\mathscr{B}(n')}(s) \right\}.$$

Note that $\mathscr{B}(n')$ is a sum of i.i.d.s of Bernoulli distribution: it is linearly increasing outside a set of exponential probability. Moreover, $\# P_{\mathscr{B}(n')}$ is linearly increasing with respect to $\mathscr{B}(n')$ in the sense of Corollary 4.4. Hence, we have $\mathcal{P}_n(\omega) \geq Kn$ outside a set of exponentially decaying probability. Fixing n, m such that $0.5Kn \leq 2^m \leq Kn$, let $\mathcal{E}$ be an equivalence classes of n-step paths made by the v-pivoting *at the first* 2^m *pivotal times.* Suppose that $\# \mathcal{P}_n(\mathcal{E}) \geq 2^m$ and label their elements as

$$\mathcal{P}_n(\mathcal{E}) := \{ i(1) < \ldots < i(2^m) < \ldots \}.$$

For $\omega \in \mathcal{E}$ we define

$$(x_{2k-1}, x_{2k}) := (\omega_{i(k)+2M_0}\, o, \omega_{i(k)+2M_0+1}\, o).$$

for $k = 1, \ldots, 2^m$. Note that x_{2k-1} is an endpoint of $\Upsilon(\beta_{i(k)})$ and x_{2k} is an endpoint of $\Upsilon(\gamma_{i(k)})$. We also set $x_0 = o$ and $x_{2\cdot 2^m+1} = \omega_n\, o$. We observe the following:

(1) $d(x_i, x_{i+1})$ is uniform in the equivalence class $\mathcal{E}$ if $i \not\equiv 1 \mod 4$. Moreover, $d(x_1, x_2)$, ..., $d(x_{2\cdot 2^m - 1}, x_{2\cdot 2^m})$ are i.i.d. with infinite second moment.
(2) For any $i < j < k$, x_i and x_k are endpoints of a D_0-aligned sequence of Schottky segments, one of whose endpoint is x_j. By Proposition 2.7, we have $(x_i, x_k)_{x_j} < E_0$ always.
(3) For any $i < j < k \leq i' < j' < k'$, $(x_i, x_k)_{x_j}$ and $(x_{i'}, x_{k'})_{x_{j'}}$ are independent.

From these ingredients, we can deduce a contradiction with the convergence in law. Since the proof is already given in [Cho21a, Section 6.1], we only sketch the idea. First observe the equality

$$\begin{aligned} d(o, \omega_n\, o) = &\underbrace{\sum_{i=1}^{2^m+1} d(x_{2i-2}, x_{2i-1})}_{I_1} + \underbrace{\sum_{i=1}^{2^m} d(x_{2i-1}, x_{2i})}_{I_2} \\ &- 2\underbrace{\sum_{l=0}^{m}\sum_{k=1}^{2^{m-l}} (x_{2^l(2k-2)}, x_{2^l\cdot 2k})_{x_{2^l(2k-1)}}}_{I_3} - 2\cdot\underbrace{(x_0, x_{2\cdot 2^m+1})_{x_{2\cdot 2^m}}}_{I_4}. \end{aligned}$$

Here, the third term I_3 is composed of sums of 2^{m-l} independent RVs bounded by E_0. Using the estimation of the variance and Chebyshev's inequality, one can deduce that

$$\mathbb{P}\left(|I_3 - \mathbb{E}[I_3 \,|\, \mathcal{E}]| > 800E_0 \cdot 2^{m/2}\right) \le 1/2000.$$

Meanwhile, I_1 is constant on $\mathcal{E}$ and I_4 is bounded by E_0. At the moment, we bring an independent equivalence class $\dot{\mathcal{E}}$ and define $\dot{I}_1, \ldots, \dot{I}_4$ in a similar manner; there is only slight chance that one of $\mathcal{E}$, $\dot{\mathcal{E}}$ have less than 2^m pivotal times, due to our choice of K. We then compare $I_1 + \mathbb{E}[I_3 \,|\, \mathcal{E}]$ and $\dot{I}_1 + \mathbb{E}[\dot{I}_3 \,|\, \dot{\mathcal{E}}]$. Since the situation is symmetric, the former will win or tie with the latter for probability at least 0.5. Now for a combination $(\mathcal{E}, \dot{\mathcal{E}})$ falling into such event, we now compare I_2 and $\dot{I}_2$; since $I_2 - \dot{I}_2$ is a sum of 2^m i.i.d.s of *symmetric* distribution with infinite second moment, for any $K' > 0$ we have

$$\mathbb{P}(I_2 - \dot{I}_2 \ge K' 2^{m/2}) \ge 1/5$$

for sufficiently large m. Combining all these, for arbitrary $K' > 10000E_0$, $d(o, \omega_n\, o) - d(o, \dot{\omega}_n o) \ge 0.5K' 2^{m/2} \ge 0.25K'\sqrt{n}$ for probability at least $1/10 - 1/500$ for sufficiently large n. However, this cannot happen for arbitrary $K' > 0$ if $\frac{1}{\sqrt{n}}[d(o, \omega_n\, o) - d(o, \dot{\omega}_n o)]$ converged in law. Hence, $\frac{1}{\sqrt{n}} d(o, \omega_n\, o)$ cannot converge in law even after suitable translation.

Let us now deduce the contradiction from the convergence in law of translation length. We gather all sample paths with at least 2^{m+1} pivotal times till n, where $m = \lfloor \log_2 Kn \rfloor - 1$; this misses only a set of probability less than $K2^{-n/K}$. At the moment, we consider the usual pivoting at the first and the last 2^{m-2} pivotal times and the v-pivoting at the intermediate pivotal times to construct an equivalence class $\mathcal{E}$. On $\mathcal{E}$, we have $\omega \in S_k^{\dagger}$ holds for some $k \le 2^{m-1}$ with probability at least $1 - (8/N_0)^{2^{m-2}}$ by Lemma 4.6. We freeze such choices for the usual pivoting at the first and the last 2^{m-2} pivotal times, and freeze some more choices for the v-pivoting at some intermediate pivotal times, to leave the freedom of 2^m v-pivotal choices at the intermediate pivotal times $i(1) < \ldots < i(2^m)$. On the finer equivalence class $\mathcal{E}_1$ after this freezing, let us define x_i's as

$$x_{2^{m+1}k+2l-1} := \omega_{i(l)+2M_0}\, o, \;\; x_{2^{m+1}k+2l} := \omega_{i(l)+2M_0+M}\, o \quad (k \in \mathbb{Z}, l = 1, \ldots, 2^m).$$

Then as before, we have that $(x_i, x_k)_{x_j} \leq E_0$ for all $i < j < k$ Here, note that $d(x_0, x_1) = d(x_{2^{m+1}}, x_{2^{m+1}+1}) = \ldots$ is constant over $\mathcal{E}_1$, since it only depends on the pivotal choices that we have already frozen. We have that

$$\tau(\omega_n) = \underbrace{\sum_{i=1}^{2^m} d(x_{2i-2}, x_{2i-1})}_{I_1} + \underbrace{\sum_{i=1}^{2^m} d(x_{2i-1}, x_{2i})}_{I_2} \underbrace{-2\sum_{l=0}^{m}\sum_{k=1}^{2^{m-l}} (x_{2^l(2k-2)}, x_{2^l\cdot 2k})_{x_{2^l(2k-1)}}}_{I_3} + I_4,$$

where

$$I_4 := \lim_k \frac{1}{k}\sum_{l=1}^{k-1} (x_0, x_{(l+1)2^{m+1}})_{x_{l2^{m+1}}}$$

is bounded by E_0. We can now deal with I_1, I_2 and I_3 just as we did for displacement: for independent and identical random walks ω and $\dot{\omega}$ generated by μ, we have

$$\begin{aligned} I_1 + \mathbb{E}[I_3|\mathcal{E}] &\geq \dot{I}_1 + \mathbb{E}[\dot{I}_3|\dot{\mathcal{E}}], \\ I_2 - \dot{I}_2 &\geq K' 2^{m/2}, \\ |I_3 - \mathbb{E}[I_3|\mathcal{E}]|, |\dot{I}_3 - \mathbb{E}[\dot{I}_3|\dot{\mathcal{E}}]| &\leq 800 E_0 2^{m/2} \end{aligned}$$

for probability at least $1/10 - 1/1000$. This implies that $\tau(\omega_n) - \tau(\dot{\omega}_n) \geq 0.5K' 2^{m/2}$ for probability at least $1/11$ for $K' > 10000E_0$ and sufficiently large n, leading to a contradiction. □

In the above proof, it is not enough to check that the intermediate progresses made by v_i's are visible in the entire progress and results in the spreading, as $[d(o, \omega_n\, o) - c_n]/\sqrt{n}$ may converge in law to an RV with infinite second moment. It really matters to precisely compare the effect by v_i's and remove the other effect, which is done by comparing independently chosen equivalence classes and working on them.

4.5. **Pivoting and repulsion among independent random walks.** In this subsection, we temporarily consider two random walks $\omega^{(1)}$, $\omega^{(2)}$ generated by non-elementary measures $\mu^{(1)}$, $\mu^{(2)}$. By adapting Proposition 2.10, we can assert the following. Fixing a constant $N_0 > 410$, there exists $K_0, M_0 > 0$ that satisfy Inequality 2.1 (recall the other constants there) and Schottky sets $S^{(1)} \subseteq (\operatorname{supp} \mu^{(1)})^{M_0}$, $S^{(2)} \subseteq (\operatorname{supp} \mu^{(2)})^{M_0}$ of cardinality at least N_0.

We now fix isometries $(w_j^{(t)})_{j=0}^{\infty}$, $(v_j^{(t)})_{j=1}^{\infty}$ for $t = 1, 2$ in G and draw choices $s^{(1)} = (\alpha_j^{(1)}, \beta_j^{(1)}, \gamma_j^{(1)}, \delta_j^{(1)})_{j=1}^{n} \in S^{4n}$ and $s^{(2)} = (\alpha_j^{(2)}, \beta_j^{(2)}, \gamma_j^{(2)}, \delta_j^{(2)})_{j=1}^{n} \in \check{S}^{4n}$ independently. We can then define the sets $P_n^{(1)}$, $P_n^{(2)}$ of pivotal times on the words

$$\begin{aligned} w^{(1)} &= w_0^{(1)} a_1^{(1)} b_1^{(1)} v_1^{(1)} c_1^{(1)} d_1^{(1)} \cdots a_n^{(1)} b_n^{(1)} v_n^{(1)} c_n^{(1)} d_n^{(1)} w_n^{(1)}, \\ w^{(2)} &= w_0^{(2)} a_1^{(2)} b_1^{(2)} v_1^{(2)} c_1^{(2)} d_1^{(2)} \cdots a_n^{(2)} b_n^{(2)} v_n^{(2)} c_n^{(2)} d_n^{(2)} w_n^{(2)}, \end{aligned}$$

respectively. Let $\mathcal{E}^{(1)} \times \mathcal{E}^{(2)}$ be the product of equivalence classes made by pivoting on $w^{(1)}$ and $w^{(2)}$, respectively. Let $P_n(\mathcal{E}^{(t)}) = \{i^{(t)}(1) < \ldots < i^{(t)}(M^{(t)})\}$ for $t = 1, 2$.

This time, we want not only that $w^{(t)}$ and $(w^{(t)})^{-1}$ heads in different directions for $t = 1, 2$, but all 4 directions made by $w^{(1)}$, $w^{(2)}$, $(w^{(1)})^{-1}$, $(w^{(2)})^{-1}$ are distinct. For this purpose, we define:

$$\begin{aligned}
\varphi_{k;inner}^{(t)} &:= (w^{(t)})^{-}_{i^{(t)}(k),2} && = w_0^{(t)} \cdots w_{i^{(t)}(k)-1}^{(t)},\\
\varphi_{k;outer}^{(t)} &:= (w^{(t)})^{-}_{i^{(t)}(k),1} && = w_0^{(t)} \cdots w_{i^{(t)}(k)-1}^{(t)} a_{i^{(t)}(k)},\\
\phi_{k;inner}^{(t)} &:= (w^{(t)})^{-1}(w^{(t)})^{+}_{i^{(t)}(M^{(t)}-k+1,0} && = (w_n^{(t)})^{-1} \cdots (v_{i^{(t)}(M^{(t)}-k+1}^{(t)})^{-1},\\
\phi_{k;outer}^{(t)} &:= (w^{(t)})^{-1}(w^{(t)})^{+}_{i^{(t)}(M^{(t)}-k+1,1} && = (w_n^{(t)})^{-1} \cdots (v_{i^{(t)}(M^{(t)}-k+1}^{(t)})^{-1}(b_{i^{(t)}(M^{(t)}-k+1}^{(t)})^{-1}.
\end{aligned}$$

We also pinpoint Schottky axes

$$\begin{aligned}
\Upsilon_{t;front} &:= \Upsilon(\alpha_{i^{(t)}(k)}^{(t)}) && = \varphi_{k;inner}^{(t)} \Gamma(\alpha_{i^{(t)}(k)}^{(t)}),\\
\Upsilon_{t;back} &:= (w^{(t)})^{-1}\, \Upsilon(\beta_{i^{(t)}(M^{(t)}-k+1)}^{(t)}) && = \phi_{k;outer}^{(t)} \Gamma(\beta_{i^{(t)}(M^{(t)}-k+1)}^{(t)}),
\end{aligned}$$

It should be noted that $\varphi_{k;inner}^{(t)}$ depends on the choices $\alpha_{i^{(t)}(1)}^{(t)}, \ldots, \gamma_{i^{(t)}(k-1)}^{(t)}$ and not on the later pivotal choices. $\varphi_{k;outer}^{(t)}$ depends on one additional factor, namely, $\alpha_{i^{(t)}(k)}^{(t)}$. $\phi_{k;inner}^{(t)}$ and $\phi_{k;outer}^{(t)}$ also have analogous dependence on the last pivotal choices.

Let us now choose $t, t' \in \{1, 2\}$. We consider three cases:

(1) front-front repulsion: let us assume $t < t'$, i.e., $t = 1$ and $t' = 2$, without loss of generality. We define:

$$\begin{aligned}
\tilde{S}_k^{(t;t'),front} &:= \left\{ \alpha_{i^{(t)}(k)}^{(t)} \in S^{(t)} : \left(\varphi_{k;inner}^{(t')} o, \Upsilon_{t;front} \right) \text{ is } K_0\text{-aligned} \right\},\\
\tilde{S}_k^{(t';t),front} &:= \left\{ \alpha_{i^{(t')}(k)}^{(t')} \in S^{(t')} : \left(\varphi_{k;outer}^{(t)} o, \Upsilon_{t';front} \right) \text{ is } K_0\text{-aligned} \right\}.
\end{aligned}$$

If $\alpha_{i^{(t)}(k)}^{(t)} \in \tilde{S}_k^{(t;t'),front}$ and $\alpha_{i^{(t')}(k)}^{(t')} \in \tilde{S}_k^{(t';t),front}$, then $(\bar{\Upsilon}_{t';front}, \Upsilon_{t;front})$ is D_0-aligned due to Lemma 2.5.

(2) back-back repulsion: we again assume $t < t'$. We define:

$$\begin{aligned}
\tilde{S}_k^{(t;t'),back} &:= \left\{ \beta_{i^{(t)}(M^{(t)}-k+1)}^{(t)} \in S^{(t)} : \left(\Upsilon_{t;back}, \varphi_{k;inner}^{(t')} o \right) \text{ is } K_0\text{-aligned} \right\},\\
\tilde{S}_k^{(t';t),back} &:= \left\{ \beta_{i^{(t')}(M^{(t')}-k+1)}^{(t')} \in S^{(t')} : \left(\Upsilon_{t';back}, \varphi_{k;outer}^{(t)} o \right) \text{ is } K_0\text{-aligned} \right\}.
\end{aligned}$$

If $\beta_{i^{(t)}(M^{(t)}-k+1)}^{(t)} \in \tilde{S}_k^{(t;t'),back}$ and $\beta_{i^{(t')}(M^{(t')}-k+1)}^{(t')} \in \tilde{S}_k^{(t';t),back}$, then $(\bar{\Upsilon}_{t';back}, \Upsilon_{t;back})$ is D_0-aligned due to Lemma 2.5.

(3) front-back repulsion: this time we do not assume $t < t'$. We define:

$$\tilde{S}_k^{(t'\nearrow t)} := \left\{ \alpha^{(t)}_{i^{(t)}(k)} \in S^{(t)} : \left(\phi^{(t')}_{k;inner} o, \Upsilon_{t;front}\right) \text{ is } K_0\text{-aligned} \right\},$$
$$\tilde{S}_k^{(t'\swarrow t)} := \left\{ \beta^{(t')}_{i^{(t')}(M^{(t')}-k+1)} \in S^{(t')} : \left(\Upsilon_{t';back}, \varphi^{(t)}_{k;outer} o\right) \text{ is } K_0\text{-aligned} \right\}.$$

If $\alpha^{(t)}_{i^{(t)}(k)} \in \tilde{S}_k^{(t'\nearrow t)}$ and $\beta^{(t')}_{i^{(t')}(M^{(t)}-k+1)} \in \tilde{S}_k^{(t'\swarrow t)}$, then $(\Upsilon_{t';back}, \Upsilon_{t;front})$ is D_0-aligned due to Lemma 2.5.

Finally, for each $t \in \{1, 2\}$ we define

$$\tilde{S}_k^{(t),front} := \left(\cap_{t'\neq t} \tilde{S}_k^{(t';t),front}\right) \cap \left(\cap_{t'\in\{1,2\}} \tilde{S}_k^{(t'\nearrow t)}\right),$$
$$\tilde{S}_k^{(t),back} := \left(\cap_{t'\neq t} \tilde{S}_k^{(t';t),back}\right) \cap \left(\cap_{t'\in\{1,2\}} \tilde{S}_k^{(t'\swarrow t)}\right).$$

Lemma 4.9. *Let $1 \le k \le \min(M^{(1)}/2, M^{(2)}/2)$. Suppose that $s^{(1)} = (\alpha^{(1)}_{i^{(1)}(l)}, \beta^{(1)}_{i^{(1)}(l)}, \gamma^{(1)}_{i^{(1)}(l)})_{l=1}^{M^{(1)}} \in \mathcal{E}_n^{(1)}$ and $s^{(2)} = (\alpha^{(2)}_{i^{(2)}(l)}, \beta^{(2)}_{i^{(2)}(l)}, \gamma^{(2)}_{i^{(2)}(l)})_{l=1}^{M^{(2)}} \in \mathcal{E}_n^{(2)}$ satisfy*

$$\alpha^{(1)}_{i^{(1)}(k)} \in \tilde{S}_k^{(1),front}, \quad \beta^{(1)}_{i^{(1)}(M^{(1)}-k+1)} \in \tilde{S}_k^{(1),back},$$
$$\alpha^{(2)}_{i^{(2)}(k)} \in \tilde{S}_k^{(2),front}, \quad \beta^{(2)}_{i^{(2)}(M^{(2)}-k+1)} \in \tilde{S}_k^{(2),back}.$$

Then $w^{(1)}$ and $w^{(2)}$ are contracting isometries that generate a free group of order 2. Moreover, the orbit map is a quasi-isometric embedding of $\langle w^{(1)}, w^{(2)}\rangle$ into a quasi-convex subset of X.

The proof is not different from the one for Lemma 4.5 so we will only sketch it. We first observe that $(\Upsilon_{t;back}, \Upsilon_{t';front})$ are K_0-aligned for all t, t', as well as $(\bar{\Upsilon}_{t;front}, \Upsilon_{t';front})$ and $(\bar{\Upsilon}_{t;back}, \Upsilon_{t';back})$ for $t \neq t'$. Now consider a word $r = w^{(1)}(w^{(2)})^{-1}$ for an example. Then we note that

$$\left(o,\ \Upsilon_{1;front},\ w^{(1)}\,\Upsilon_{1;back},\ w^{(1)}\bar{\Upsilon}_{2;back},\ w^{(1)}(w^{(2)})^{-1}\bar{\Upsilon}_{2;front}, ro\right)$$

is a subsequence of D_0-aligned sequence, hence D_1-aligned. This implies that

$$\begin{aligned} d(o, ro) &\ge \left(d(\varphi^{(1)}_{k;inner} o, w^{(1)}\phi^{(1)}_{k;inner} o) + d(w^{(1)}\phi^{(2)}_{k;inner} o, w^{(1)}(w^{(2)})^{-1}\varphi^{(2)}_{k;inner} o)\right) - 2E_0 \cdot 4 \\ &\ge \left(d(\varphi^{(1)}_{k;inner} o, w^{(1)}\phi^{(1)}_{k;inner} o) + d(w^{(2)}\phi^{(2)}_{k;inner} o, \varphi^{(2)}_{k;inner} o)\right) - 2E_0 \cdot 4. \end{aligned}$$

Likewise, any word r of letters $w^{(1)}$ and $w^{(2)}$ has displacement at least $C|r|$, where $|r|$ is the word length and

$$C = \min\left\{ d(\varphi^{(t)}_{k;inner} o, w^{(t)}\phi^{(t)}_{k;inner} o), d(w^{(t)}\phi^{(t)}_{k;inner} o), \varphi^{(t)}_{k;inner} o) : t = 1, 2\right\} - 4E_0 > 0.$$

Furthermore, thanks to the alignment, we have that $[o, ro]$ passes through the E_0-neighborhoods of o, $w^{(1)}o$, $w^{(1)}(w^{(2)})^{-1}o$ and ro. In other words, $[o, ro]$ lies within the K-neighborhood of $\{go : g \in \langle w^{(1)}, w^{(2)}\rangle\}$ where $K = \max\{d(o, w^{(1)}o), d(o, w^{(2)}o)\} + 2E_0$.

We now estimate the probability that this happens. For notational purpose, we temporarily denote by

$$s_k = \begin{pmatrix} \alpha^{(1)}_{i^{(1)}(k)}, \ \beta^{(1)}_{i^{(1)}(k)}, \gamma^{(1)}_{i^{(1)}(k)}, \ \alpha^{(1)}_{i^{(1)}(M^{(1)}-k+1)}, \ \beta^{(1)}_{i^{(1)}(M^{(1)}-k+1)}, \ \gamma^{(1)}_{i^{(1)}(M^{(1)}-k+1)}, \\ \alpha^{(2)}_{i^{(2)}(k)}, \ \beta^{(2)}_{i^{(2)}(k)}, \ \gamma^{(2)}_{i^{(2)}(k)}, \ \alpha^{(1)}_{i^{(2)}(M^{(2)}-k+1)}, \ \beta^{(2)}_{i^{(2)}(M^{(2)}-k+1)}, \ \gamma^{(2)}_{i^{(2)}(M^{(2)}-k+1)}, \end{pmatrix}$$

a choice in $\tilde{S}^{(1)}_{i^{(1)}(k)} \times \tilde{S}^{(1)}_{i^{(1)}(M^{(1)}-k+1)} \times \tilde{S}^{(2)}_{i^{(2)}(k)} \times \tilde{S}^{(1)}_{i^{(2)}(M^{(2)}-k+1)}$. Given a partial choice

$$\bar{s} = (s_1, \ldots, s_{k-1}) \in \prod_{t \in \{1,2\}, 1 \le l \le k-1} \tilde{S}^{(t)}_{i^{(t)}(k)} \times \tilde{S}^{(t)}_{i^{(t)}(M^{(t)}-k+1)},$$

we define

$$\tilde{S}^{\dagger}_k := \left\{ s_k : \alpha_{i^{(t)}(k)} \in \tilde{S}^{(t),front}_k(\bar{s}, s_k), \ \beta_{i^{(t)}(M^{(t)}-k+1)} \in \tilde{S}^{(t),back}_k(\bar{s}, s_k) \text{ for } t \in \{1,2\} \right\}.$$

Then we have the following:

Lemma 4.10. *For each* $1 \le k \le \min(M^{(1)}/2, M^{(2)}/2)$*, we have*

$$\#\tilde{S}^{\dagger}_k \ge \left((\#S^{(1)})^6 - (12\#S^{(1)})^5 \right) \left((\#S^{(2)})^6 - (12\#S^{(2)})^5 \right).$$

Proof. We first pick appropriate $\gamma^{(t)}_{i^{(t)}(k)}$'s and $\gamma^{(t)}_{i^{(t)}(M^{(t)}-k+1)}$'s that satisfy Inequality 4.2. Fixing such choices, we then pick $\beta^{(t)}_{i^{(t)}(k)}$'s that satisfy Inequality 4.4. Now, we first pick $\alpha^{(1)}_{i^{(1)}(k)}$ that satisfy Inequality 4.5 *and the conditions for* $\tilde{S}^{(1),front}_k$. The latter conditions are that

$$\begin{aligned} &\left((\varphi^{(1)}_{k;inner})^{-1} \varphi^{(2)}_{k;inner} o, \Gamma(\alpha^{(1)}_{i^{(1)}(k)}) \right) \text{ is } K_0\text{-aligned}, \\ &\left((\phi^{(1)}_{k;inner})^{-1} \varphi^{(1)}_{k;inner} o, \Gamma(\alpha^{(1)}_{i^{(1)}(k)}) \right) \text{ is } K_0\text{-aligned}, \\ &\left((\phi^{(1)}_{k;inner})^{-1} \varphi^{(2)}_{k;inner} o, \Gamma(\alpha^{(1)}_{i^{(1)}(k)}) \right) \text{ is } K_0\text{-aligned}; \end{aligned}$$

all of these conditions are determined by the choice of $\bar{s}$ and do not depend on other coordinates of s_k. After deciding the choice of $\alpha^{(1)}_{i^{(1)}(k)}$ we pick valid $\alpha^{(2)}_{i^{(2)}(k)}$; this time it should satisfy Inequality 4.5 and the conditions for $\tilde{S}^{(2),front}_k$, which depend on $\bar{s}$ and $\alpha^{(1)}_{i^{(1)}(k)}$.

Next we move on to choosing $\beta^{(1)}_{i^{(1)}(M^{(1)}-k+1)}$. It should satisfy Inequality 4.4 and the conditions for $\tilde{S}^{(1),back}_k$; these depend on $\bar{s}$ and $\alpha^{(1)}_{i^{(1)}(k)}$, $\alpha^{(2)}_{i^{(2)}(k)}$. We then pick appropriate $\beta^{(2)}_{i^{(2)}(M^{(2)}-k+1)}$ and then the remaining choices.

Following this order, we obtain at least

$$\begin{aligned} &\left(\#S^{(1)} - 1 \right)^4 \left(\#S^{(2)} - 1 \right)^4 \left(\#S^{(1)} - 4 \right)^2 \left(\#S^{(2)} - 4 \right)^2 \\ &\ge \left((\#S^{(1)})^6 - (12\#S^{(1)})^5 \right) \left((\#S^{(2)})^6 - (12\#S^{(2)})^5 \right) \end{aligned}$$

valid choices for s_k thanks to the properties of Schottky sets $S^{(1)}$, $S^{(2)}$. □

Recall that we have implemented pivotal times for random walks in [Cho22b, Section 4.3], which is pretty much repeated in the proof of Proposition 4.8. Given this, by combining Lemma 4.9 and Lemma 4.6, we deduce the following corollary:

Corollary 4.11. *Let (X, G, o) be as in Convention 1.1 and $\omega^{(1)}, \omega^{(2)}$ be two independent random walks generated by a non-elementary measure μ on G. Then there exists $K > 0$ such that the following holds outside a set of probability $Ke^{-n/K}$. The n-th step isometries $\omega_n^{(1)}$, $\omega_n^{(2)}$ arising from two random walks generate a free group of order 2. Moreover, the orbit map is a quasi-isometric embedding of $\langle \omega_n^{(1)}, \omega_n^{(2)} \rangle$ into a quasi-convex subset of X.*

It is also not difficult to consider k independent random walks; the arguments are identical. Hence, we conclude Theorem D.

5. Counting problem

We first begin with a quantitative version of the main theorem in [Cho21b].

Theorem 5.1 (Translation length grows linearly)**.** *For each $\lambda > 1$, there exists $\lambda_0 > 0$ satisfying the following. Let G be a finitely generated non-elementary subgroup of* $\mathrm{Isom}(X)$ *and $S' \subseteq G$ be a finite symmetric generating set.*

Then there exists a set $S'' \supseteq S'$ of G with $\#S'' \le (1+\lambda)\#S' + \lambda_0$ such that

$$\frac{\#\{g \in B_{S''}(n) : g \text{ is not contracting or } \tau_X(g) \le Ln\}}{\#B_{S''}(n)} \le Ke^{-n/K}$$

holds for some $L > K$.

Our strategy is to add Schottky isometries to S'. We encounter one technicality: the K_0-Schottky set S that we have in hand can never be symmetric. Hence, in the following construction, we should allow choosing $\alpha_i, \beta_i, \gamma_i, \delta_i$ from $S \cup \check{S}$ for the pivotal time construction, where

$$\check{S} := \{s^{-1} : s \in S\} = \{(a_{M_0}^{-1}, \ldots, a_1^{-1}) : (a_1, \ldots, a_{M_0} \in S\}.$$

Lemma 5.2. *Let $s_i \in S$ and $\epsilon_i \in \{\pm 1\}$ for $i = 1, \ldots, k$. Suppose that there does not exist i such that $s_i = s_{i+1}$ and $\epsilon_i \epsilon_{i+1} = -1$. Then:*

(1) the sequence

$$\left(\Gamma(s_1^{\epsilon_1}), \quad \Pi(s_1^{\epsilon_1})\Gamma(s_2^{\epsilon_2}), \quad \ldots, \quad \Pi(s_1^{\epsilon_1}) \cdots \Pi(s_{k-1}^{\epsilon_k})\Gamma(s_k^{\epsilon_k})\right)$$

is D_0-aligned, and

(2) $\Pi(s_1^{\epsilon_1}) \cdots \Pi(s_k^{\epsilon_k})$ is not the identity element.

Proof. Note that

$$\mathrm{diam}\left(\pi_{\Gamma(s^\epsilon)}(\Pi(s^\epsilon)o) \cup o\right) = \mathrm{diam}\left(\Pi(s^\epsilon)o \cup o\right) \ge M_0/K_0 - K_0 > K_0$$

holds for each $s \in S$ and $\epsilon \in \{\pm 1\}$. This implies that

$$\operatorname{diam}\left(\pi_{\Gamma^n(s')}(\Pi(s^\epsilon)o) \cup o\right) \le K \tag{5.1}$$

holds for all n if $s \neq s'$ (Property (2)), and for $n\epsilon \le 0$ if $s = s'$ (Property (3)).

Now for each i, we have the following cases.

(1) $s_i \neq s_{i+1}$: then we have

$$\operatorname{diam}\left(\pi_{\Gamma\left(s_{i+1}^{\epsilon_{i+1}}\right)}(\Pi(s_i^{-\epsilon_i})o) \cup o\right) \le K_0, \quad \operatorname{diam}\left(\pi_{\Gamma(s_i^{\epsilon_i})}(\Pi(s_i^{\epsilon_i})o) \cup \Pi(s_i^{\epsilon_i})o\right) = 0.$$

Here, the first inequality is Inequality 5.1 and the second inequality is immediate. Hence, $\left(\Gamma(s_i^{\epsilon_i}),\, \Pi(s_i^{\epsilon_i})\Gamma(s_{i+1}^{\epsilon_{i+1}})\right)$ is D_0-aligned by Lemma 2.5.

(2) $s_i = s_{i+1}$: then $\epsilon_i = \epsilon_{i+1}$, and the above inequalities similarly hold.

This concludes the D_0-alignment. Now the nontriviality of $\Pi(s_1^{\epsilon_1}) \cdots \Pi(s_k^{\epsilon_k})$ follows from this D_0-alignment, namely,

$$d(o, \Pi(s_1^{\epsilon_1}) \cdots \Pi(s_k^{\epsilon_k})o) \ge \left[\sum_{i=1}^k d\left(o, \Pi(s_k^{\epsilon_k})o\right)\right] - 2(k-1)E_0 \ge E_0 k. \quad \square$$

This leads to the following corollary:

Corollary 5.3. *S and $\check{S}$ are disjoint. Moreover, if we define*

$$T := \{(s_1, s_2, s_3, s_4) \in \left(s_i \in S \cup \check{S}\right)^4 : s_i \neq s_{i+1}^{-1} \text{ for } i = 1, 2, 3\} \tag{5.2}$$

and the map

$$\Phi : T \to G, \quad \Phi(s_1, s_2, s_3, s_4) := \Pi(s_1)\Pi(s_2)\Pi(s_3)\Pi(s_4), \tag{5.3}$$

then f is injective.

Proof of Theorem 5.1. Let us first observe the function

$$f(x) := \frac{1}{1+\sqrt{\lambda}} \left(\frac{\sqrt{\lambda}}{x}\right)^x \left(\frac{1}{1-x}\right)^{1-x}.$$

We have $\lim_{x \to 0+} f(x) = 1/(1+\sqrt{\lambda}) < 0.5$ so there exists $0 < \epsilon_1 < 1/3$ such that $f(\epsilon_1) \le 1/2$. We then set

$$\lambda_0 = \left\lceil \left(24\sqrt{\lambda}\right)^4 \left(2^{20/\epsilon_1} + \left(\frac{1}{\sqrt{\lambda}-1}\right)^4\right) \right\rceil.$$

Our choice of λ_0 satisfies that:

$$1 - \frac{12}{\sqrt[4]{\lambda_0}} \ge 1 - \frac{1}{2\sqrt{\lambda}/(\sqrt{\lambda}-1)} = \frac{1+1/\sqrt{\lambda}}{2} \ge 1/\sqrt{\lambda}, \tag{5.4}$$

$$\lambda_0/\sqrt{\lambda} \ge 12 \cdot 2^{20/\epsilon_1}. \tag{5.5}$$

Given S', let S be the K_0-Schottky set with cardinality $\lfloor \frac{1}{2}\sqrt[4]{\lambda \# S' + \lambda_0} \rfloor$. We then define T and $\Phi : T \to G$ as in Equation 5.2 and 5.3. We then have

$$N_0 := \#\Phi(T) = \#T = \left(2\left\lfloor \frac{1}{2}\sqrt[4]{\lambda \# S' + \lambda_0}\right\rfloor\right)\left(2\left\lfloor \frac{1}{2}\sqrt[4]{\lambda \# S' + \lambda_0}\right\rfloor - 1\right)^3 \leq \lambda \# S' + \lambda_0$$

and

$$\begin{aligned} N_0 &\geq \left(\sqrt[4]{\lambda \# S' + \lambda_0} - 3\right)^4 \geq (\lambda \# S' + \lambda_0)\left(1 - \frac{12}{\sqrt[4]{\lambda \# S' + \lambda_0}}\right) \\ &\geq \sqrt{\lambda}\# S' + \lambda_0/\sqrt{\lambda} \geq \sqrt{\lambda}\# S' + 8 \cdot 2^{20/\epsilon_1}. \end{aligned}$$

Here, we used Inequality 5.4 and 5.5 at the second and the third inequalities, respectively.

We consider the simple random walk on $S' \cup \Phi(T)$. We have

$$\mu = \alpha \mu_{\Phi(T)} + (1 - \alpha)\nu,$$

where $\mu_{\Phi(T)}$ is the uniform measure on $\Phi(T)$ and ν is the uniform measure on the remaining choices. Here, note that $\alpha \geq \sqrt{\lambda}/(1 + \sqrt{\lambda})$. As in the proof of Proposition 4.8, we consider:

- Bernoulli RVs ρ_i with $\mathbb{P}(\rho_i = 1) = \alpha$ and $\mathbb{P}(\rho_i = 0) = 1 - \alpha$,
- η_i' with the law $\mu_{\Phi(T)}$, and
- ν_i with the law ν,

all independent, and define

$$g_{k+1} = \begin{cases} \nu_k & \text{when } \rho_k = 0, \\ \eta_k' & \text{when } \rho_k = 1. \end{cases}$$

Then $(g_i)_{i=1}^{\infty}$ has the law μ^{∞}. We now define Ω to be the ambient probability space on which the above RVs are all measurable. We will denote an element of Ω by ω. We also fix

- $\omega_k := g_1 \cdots g_k$,
- $\mathscr{B}(k) := \sum_{i=0}^{k} \rho_i$, i.e., the number of the Schottky slots till k, and
- $\vartheta(i) := \min\{j \geq 0 : \mathscr{B}(j) = i\}$, i.e., the i-th Schottky slot.

Let us estimate the probability that $\mathscr{B}(n-1) \leq \epsilon_1 n$. Since $\mathscr{B}(n-1)$ is greater in distribution than the sum of n independent Bernoulli distribution with expectation $\sqrt{\lambda}/(1+\sqrt{\lambda})$, we have

$$\mathbb{P}\left(\mathscr{B}(n-1) \leq \epsilon_1 n\right) \leq \sum_{i=0}^{\epsilon_1 n} \binom{n}{i} \left(\frac{\sqrt{\lambda}}{1+\sqrt{\lambda}}\right)^i \left(\frac{1}{1+\sqrt{\lambda}}\right)^{n-i}.$$

Since $\epsilon_1/(1-\epsilon_1) \leq 1/\sqrt{\lambda}$, the term $a_i = \binom{n}{i}\left(\frac{\sqrt{\lambda}}{1+\sqrt{\lambda}}\right)^i \left(\frac{1}{1+\sqrt{\lambda}}\right)^{(1-\epsilon_1)n}$ is monotonically increasing for $i = 0, \ldots, \epsilon_1 n$. Hence, the probability is bounded by

$$\epsilon_1 n \cdot \binom{n}{\epsilon_1 n} \left(\frac{\sqrt{\lambda}}{1+\sqrt{\lambda}}\right)^{\epsilon_1 n} \left(\frac{1}{1+\sqrt{\lambda}}\right)^{(1-\epsilon_1)n}.$$

The growth rate of this term is $f(\epsilon_1)$, which is smaller than 1/2. Hence, we can conclude that

$$\mathbb{P}\left(\mathscr{B}(n-1) \leq \epsilon_1 n\right) \leq \frac{C}{2^n}$$

for some $C > 0$.

Now given the choices of $\{\rho_i\}_i$ that gives $\mathscr{B}(n-1) \geq \epsilon_1 n$, we further fix the values of ν_i's. At the moment, we define $\eta_i := (\alpha_i, \beta_i, \gamma_i, \delta_i) \in T$ such that $\Phi(\alpha_i, \beta_i, \gamma_i, \delta_i) = \eta'_{\vartheta(i)+1}$; note that the correspondence $\eta_i \leftrightarrow \eta'_{\vartheta(i)+1}$ is one-to-one. Then η_i's are chosen with the uniform measure $T^{\mathscr{B}(n-1)}$. We first define

$$w_i := g_{\vartheta(i-1)+2} \cdots g_{\vartheta(i)}.$$

We also set $w^{(n)} := g_{\vartheta(\mathscr{B}(n-1))+2} \cdots g_n$ and $a_i = \Pi(\alpha_i), \ldots, d_i = \Pi(\delta_i)$. Then we have

$$\begin{aligned}\omega_n &= w_0 \nu_{\vartheta(1)} w_2 \cdots \nu_{\vartheta(\mathscr{B}(n-1))} w^{(n)} \\ &= w_0 a_1 b_1 c_1 d_1 w_1 \cdots a_{\mathscr{B}(n-1)} b_{\mathscr{B}(n-1)} c_{\mathscr{B}(n-1)} d_{\mathscr{B}(n-1)} w^{(n)}.\end{aligned}$$

In this setting, we define the set of pivotal times as in [Cho22b, Subsection 4.1]. A slight difference here is that $(\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^{\mathscr{B}(n-1)}$ is chosen with the uniform measure on $T^{\mathscr{B}(n-1)}$, not $S^{4\,\mathscr{B}(n-1)}$. This affects the lemmata in [Cho22b, Subsection 4.1] as follows.

- For choices $(\alpha_i, \beta_i, \gamma_i, \delta_i) \in T$, [Cho22b, Observation A.1] still holds thanks to Lemma 5.2 and Lemma 4.1 also holds.
- In Lemma 4.2, we first pick $\delta_i \in S \cup \check{S}$, and then $\gamma_i \in (S \cup \check{S}) \setminus \{\delta_i\}^{-1}$, and then $\beta_i \in (S \cup \check{S}) \setminus \{\gamma_i^{-1}\}$, and then $\alpha_i \in (S \cup \check{S}) \setminus \{\beta_i^{-1}\}$. First, there exists at most 1 candidate for δ_k that violates Condition 4.3; this rules out at most $(N_0 - 1)^3$ choices in T. Picking δ_k that satisfies Condition 4.3, Condition 4.2 and 4.4 are automatically guaranteed for any valid γ_k and β_k due to the definition of T and Lemma 5.2. Finally, there exists at most 1 candidate for α_k that violates Condition 4.5. This rules out at most $N_0(N_0 - 1)^2$ choices in T. Overall, we have

$$\mathbb{P}\left(\#P_k(s, \alpha_k, \beta_k, \gamma_k, \delta_k) = \#P_{k-1}(s) + 1\right) \geq 1 - \frac{(2N_0 - 1)(N_0 - 1)^2}{N_0(N_0 - 1)^3} \geq 1 - \frac{2}{N_0 - 1}.$$

- Similarly, in the proof of [Cho22b, Lemma 4.4], we first have

$$\mathbb{P}(\mathcal{A}|T) \geq 1 - \frac{2}{N_0 - 1}.$$

  Next, in the case of $j = 1$ we similarly set $l < m$ as the last 2 elements of $P_{k-1}(s)$. Fixing $(\alpha_k, \beta_k, \gamma_k, \delta_k) \in T$ and $\tilde{s} \in \mathcal{E}_{k-1}(s)$, we define $\tilde{A} = \tilde{A}(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \in \tilde{S}_m(s)$ as in the proof of [Cho22b, Lemma

4.4]. In other words, for $(\tilde{\alpha}_m, \bar{\beta}_m, \tilde{\gamma}_m) \in \tilde{A}$, $\bar{\beta}_m$ is now subject to

(5.6)
$$\begin{aligned}&\operatorname{diam}\left(\pi_{\Gamma^{-1}(\bar{\beta}_m)}((\tilde{w}_{m,0}^{-})^{-1}\tilde{w}_{k-1,2}^{-}a_k b_k v_k c_k d_k o) \cup o\right)\\ &= \operatorname{diam}\left(o \cup \pi_{\Gamma^{-1}(\bar{\beta}_m)}(v_m \tilde{c}_m \tilde{d}_m w_m \cdots \tilde{a}_{k-1}\tilde{b}_{k-1}v_{k-1}\tilde{c}_{k-1}\tilde{d}_{k-1}w_{k-1} \cdot a_k b_k v_k c_k d_k w_k o)\right) < K_0\end{aligned}$$

in addition to the standing condition that $\bar{\beta}_m \neq \tilde{\alpha}_m^{-1}, \tilde{\gamma}_m^{-1}$. Since the additional Condition 5.6 rules out at most 1 choice, we have the conditional expectation

$$\frac{\#[E(\tilde{s}, \tilde{S}_m) \setminus E(\tilde{s}, \tilde{A})]}{\#E(\tilde{s}, \tilde{S}_m)} \leq \frac{1}{N_0 - 2} \geq \frac{2}{N_0 - 1}.$$

This leads to the estimation

$$\begin{aligned}&\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - 1 \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s), (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right)\\ &\leq \frac{2}{N_0 - 1} \cdot \frac{2}{N_0 - 1}.\end{aligned}$$

By similar induction steps, we get

$$\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - j \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s),\, (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right) \leq \left(\frac{2}{N_0 - 1}\right)^{j+1}$$

This eventually affects Corollary 4.4.

- For the pivoting for translation length, let us compare the proportion of $S_k^{\dagger}$ in $\tilde{S}_{i(k)} \times \tilde{S}_{i(M-k+1)}$ for an equivalence class $\mathcal{E}$ with M pivotal times. Fixing valid choices for $\beta_{i(k)}, \gamma_{i(k)}, \alpha_{i(M-k+1)}$ and $\gamma_{i(M-k+1)}$, we have three constraints for $\alpha_{i(k)}$: $\alpha_{i(k)} \neq \beta_{i(k)}^{-1}$, Condition 4.5 and Condition 4.7. In other words, among at least $N_0 - 2$ choices of $\alpha_{i(k)}$ that makes $(\alpha_{i(k)}, \beta_{i(k)}, \gamma_{i(k)}) \in \tilde{S}_{i(k)}$, all choices but at most one satisfy Condition 4.7. Fixing such $\alpha_{i(k)}$, we obtain a similar estimate for $\beta_{i(M-k+1)}$ and we conclude

$$\mathbb{P}\left(\alpha_{i(k)} \in S_k^*(s), \beta_{i(M-k+1)} \in S_{M-k+1}^*(s) \text{ for some } k \leq m \,\middle|\, \mathcal{E}\right) \geq 1 - \left(\frac{2}{N_0 - 2}\right)^m.$$

Having these modifications, we now estimate

$$\mathbb{P}\left(\#P_n(\omega) \geq \epsilon_1 n/2 \,\middle|\, \mathscr{B}(\omega_n) \geq \epsilon_1 n\right).$$

If $\mathscr{B}(\omega_n) = N$, then $\#P_n(\omega)$ is greater in distribution than the sum of N i.i.d. X_i with the distribution

$$\mathbb{P}(X_i = j) = \begin{cases} 1 - \frac{2}{N_0 - 1} & \text{if } j = 1, \\ \left(1 - \frac{2}{N_0-1}\right)\left(\frac{2}{N_0-1}\right)^{-j} & \text{if } j < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

Note that

$$\mathbb{E}\left[\sqrt{\frac{2}{N_0-1}}^{X_i}\right] = \left(1-\frac{2}{N_0-1}\right)\left[\sqrt{\frac{2}{N_0-1}}+\sum_{i=1}^{\infty}\sqrt{\frac{2}{N_0-1}}^{i}\right] \le 2.1\sqrt{\frac{2}{N_0-1}}.$$

We then calculate:

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{N} X_i < \epsilon_1 n/2\right)\cdot\sqrt{\frac{2}{N_0-1}}^{\epsilon_1 n/2} &\le \mathbb{E}\left[\sqrt{\frac{2}{N_0-1}}^{\sum_{i=1}^N X_i}\right] \\ &= \prod_{i=1}^{N}\mathbb{E}\left[\sqrt{\frac{2}{N_0-1}}^{X_i}\right] \\ &\le 2.1^{\epsilon_1 n}\cdot\sqrt{\frac{2}{N_0-1}}^{\epsilon_1 n}.\end{aligned}$$

This implies that

$$\mathbb{P}\left(\sum_{i=1}^{N} X_i < \epsilon_1 n/2\right) \le 2.1^{\epsilon_1 n}\cdot\left(\frac{2}{N_0-1}\right)^{\epsilon_1 n/4} \le \left(\frac{2\cdot 20}{N_0}\right)^{\epsilon_1 n/4} \le \frac{1}{2^n}.$$

At the final stage we used $N_0 \ge 40\cdot 2^{5/\epsilon_1}$.

Now, for an equivalence class $\mathcal{E}_n$ with $P_n(\mathcal{E}_n) \ge \epsilon_1 n/2$, we know that ω is contracting with $\tau(\omega) \ge \epsilon_1 n/10$ except probability

$$\left(\frac{2}{N_0-2}\right)^{\epsilon_1 n/5} \le \frac{1}{2^n}.$$

In summary, $\mathbb{P}(\omega_n \text{ is not contracting or } \tau(\omega_n) \ge \epsilon_1 n/10) \le 1-(1/2)^n$; the number of sample paths corresponding to this event is at most $((\#S' + N_0)/2)^n$.

Meanwhile, the ball $B_n(e)$ contains all

$$\{\Pi(s_1)\cdots\Pi(s_{4n}) : s_i \in S_0, s_i \ne s_{i+1}^{-1}\}.$$

Their number is at least

$$(\sqrt[4]{\lambda\#S'+\lambda_0}-3)^{4n} \ge \left((\lambda\#S'+\lambda_0)\left(1-\frac{12}{\sqrt[4]{\lambda_0}}\right)\right)^n \ge \left((\lambda\#S'+\lambda_0)\left(\frac{1+1/\sqrt{\lambda}}{2}\right)\right)^n.$$

Since

$$\#S' + N_0 \le (1+\lambda)\#S' + \lambda_0 \le (\sqrt{\lambda}+\lambda)\#S' + \lambda_0(1+1/\sqrt{\lambda}),$$

we conclude that the growth rate of $\#B_n(e)$ is strictly greater than the growth rate of elements w in $\#B_n(e)$ such that w is not contracting or $\tau(w) \ge \epsilon_1 n/10$. □

Using a similar argument that involves pivoting for quasi-isometric embedding of k independent random walks (Lemma 4.10), we can deduce the following version of Theorem E:

Theorem 5.4. *For each $k \in \mathbb{Z}_{>0}$ and $\lambda > 1$, there exists $\lambda_0 > 0$ satisfying the following. Let G be a finitely generated non-elementary subgroup of $\mathrm{Isom}(X)$ and $S' \subseteq G$ be a finite symmetric generating set.*

Then there exists a set $S'' \supseteq S'$ of G with $\#S' \leq (1+\lambda)\#S' + \lambda_0$ such that for all k-tuples $(g_1, \ldots, g_k)$ of elements in $B_{S''}(n)$ except an exponentially decaying proportion, $\langle g_1, \ldots, g_k\rangle$ is q.i. embedded into a quasi-convex subset of X.

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