RANDOM WALKS AND CONTRACTING ELEMENTS III: OUTER SPACE AND OUTER AUTOMORPHISM GROUP

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Abstract. Continuing from [Cho22b], we study random walks on (possibly asymmetric) metric spaces using the bounded geodesic image property (BGIP) of certain isometries. As an application, we show that a generic outer automorphism of the free group of rank at least 3 has different forward and backward expansion factors. This answers a question of Handel and Mosher in [HM07b]. Together with this, we also revisit limit laws on Outer space including SLLN, CLT, LDP and the genericity of a fully irreducible outer automorphism.
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1. Introduction

This is the third in a series of articles concerning random walks on metric spaces with contracting elements. This series is a reformulation of the previous preprint [Cho22a] announced by the author. In this article, we adopt the following setting:

Convention 1.1. *Throughout, we assume that:*

- (X, d) *is a geodesic metric space, possibly with an asymmetric metric;*
- G *is a countable group of isometries of* X*, and*
- G *contains two independent isometries that exhibit the bounded geodesic image property (BGIP).*

We also fix a basepoint $o \in X$.

The main purpose of this article is to generalize the random walk theory in [Cho22b] to asymmetric metric spaces. Our main result deals with the mismatch between the forward and backward translation length of a generic isometry of X.

Theorem A (Asymmetry of a generic translation length)**.** *Let* (X, d, G) *be as in Convention 1.1. Let* ω *be the random walk generated by a non-elementary, asymptotically asymmetric measure* μ *on* G*. Then for any* $K > 0$*, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(\omega : |\tau(\omega_n) - \tau(\omega_n^{-1})| < K\right) = 0.$$

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Together with this, the limit laws discussed in [Cho22b] and [Cho22c], including SLLN, CLT, LIL, geodesic tracking and the genericity of q.i. embedded subgroups, also generalize to the current setting of asymmetric metric spaces. As an application, we obtain the following corollary:

Corollary 1.2. *Let X be Outer space of rank $N \geq 3$ and G be the corresponding outer automorphism group of the free group F_N. Let ω be an admissible random walk on G. Then for any $K > 0$, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(\omega : |\tau(\omega_n) - \tau(\omega_n^{-1})| < K\right) = 0.$$

Corollary 1.2 asserts that a generic outer automorphism is a fully irreducible that has a different expansion factor than its inverse. This has been suggested by Handel and Mosher in [HM07b]. There, they proved the asymmetry for a large class of automorphisms, namely, the class of parageometric fully irreducibles. In contrast, a companion result in [HM07a] states that for any fully irreducible element $\phi \in \mathrm{Out}(F_N)$, the expansion factor λ of ϕ and λ' of ϕ^{-1} satisfy $1/C \leq \log(\lambda/\lambda') \leq C$ for some constant C that depends on N, the rank of the free group. In other words, the translation lengths of ϕ and ϕ^{-1} are within bounded ratio (see also [AKB12]).

Together with this, we also recover Horbez's SLLN [Hor16a] and CLT [Hor18] for displacement, and Dahmani-Horbez's SLLN for translation length [DH18] on Outer space, the latter with a weaker and optimal moment condition. We also present a CLT for translation length and its converse, which seems new for Outer space. Moreover, we obtain optimal deviation inequalities on Outer space; see [Hor18] for previously known deviation inequalities. Using them, we also establish the geodesic tracking of random walks. Finally, we also discuss the exponential genericity of (atoroidal) fully irreducible automorphisms, which is a recurring theme in [MT18], [TT16] and [KMPT22]; note that we do not require moment conditions here.

In order to apply our general theory to Outer space, we crucially utilize the BGIP of fully irreducible outer automorphisms. Namely, we modify Kapovich-Maher-Pfaff-Taylor's observation ([KMPT22, Theorem 7.8]) into the following form:

Proposition 1.3. *Let $\varphi \in \mathrm{Out}(F_N)$ be a fully irreducible outer automorphism. Then the orbit $\{\varphi^i o\}_{i\in\mathbb{Z}}$ of o by φ is a BGIP axis.*

1.1. **Structure of the article.** In Section 2, we recall the notion of bounded geodesic image property (BGIP) and prove relevant lemmata. These lemmata have been used in [Cho22b] to establish subsequent alignment lemmata. Here, we rephrase them in the language of BGIP. In Section 3, we review preliminaries on the outer automorphism group and Outer space. The main theorem of this section is the BGIP of fully irreducible outer automorphisms on Outer space. In Section 4, we first generalize the limit laws discussed in [Cho22b] and [Cho22c] while pointing out subtle differences. Next, we review the notion of pivotal times and the pivoting technique that are established in [Cho22b]. Using these, we finally prove Theorem A in Subsection 4.3.

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2. Preliminaries

2.1. Asymmetric metric spaces.

Definition 2.1 (Metric space)**.** *An* (asymmetric) metric space (X, d) *is a set* X *equipped with a function* $d : X \times X \to \mathbb{R}_{\geq 0}$ *that satisfies the following:*

- *for any* $x, y \in X$*,* $d(x, y) = 0$ *if and only if* $x = y$*;*
- *(triangle inequality) for any* $x, y, z \in X$*,* $d(x, z) \leq d(x, y) + d(y, z)$*;*
- *(local symmetry) for each* $x \in X$*, there exist* $\epsilon, K > 0$ *such that* $d(y, z) \leq K d(z, y)$ *holds for* $y, z \in \{a \in X : \min(d(x, a), d(a, x)) < \epsilon\}$.

In this situation, we say that d *is a* metric *on* X*.* d *is said to be* symmetric *if* $d(x, y) = d(y, x)$ *holds for all* $x, y \in X$*. We define a symmetric metric called the* symmetrization *of* d *by*

$$d^{sym}(x, y) := d(x, y) + d(y, x).$$

We endow (X, d) *with the topology induced by* d^{sym}*.*

The diameter *of a set* $A \subseteq X$ *is defined by*

$$\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\},$$

and the (directed) distances *between sets* $A, B \subseteq X$ *are defined by*

$$d(A, B) := \inf\{d(x, y) : x \in A, y \in B\},$$
$$d^{sym}(A, B) := \inf\{d^{sym}(x, y) : x \in A, y \in B\}.$$

For $R > 0$*, the* R-neighborhood *of a set* $A \subseteq X$ *is defined by*

$$\mathcal{N}_R(A) := \{x : d^{sym}(x, A) < R\}.$$

The Hausdorff distance *between* $A, B \subseteq X$ *is defined by*

$$d_H(A, B) := \inf\{R > 0 : A \subseteq \mathcal{N}_R(B) \text{ and } B \subseteq \mathcal{N}_R(A)\}.$$

Definition 2.2 (Quasigeodesics). *A path $\gamma : I \to X$ from an interval or a set of consecutive integers I is called a K-*quasigeodesic *if*

$$\frac{1}{K}|t-s| - K \le d(\gamma(s), \gamma(t)) \le K|t-s| + K \tag{2.1}$$

*holds for all $s, t \in I$ such that $s < t$. If Inequality 2.1 holds for all $s, t \in I$, we say that γ is a K-*bi-quasigeodesic.

A metric space X is said to be geodesic *if every ordered pair of points can be connected by a geodesic, i.e., for every $x, y \in X$ there exists a geodesic $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$.*

We will frequently use Inequality 2.1 in the following form. For any points p, q on a K-bi-quasigeodesic γ, we have

$$\operatorname{diam}\left(\gamma^{-1}(p) \cup \gamma^{-1}(q)\right) \le K d(p, q) + K^2 \tag{2.2}$$

and

$$d(q, p) \le K \operatorname{diam}\left(\gamma^{-1}(p) \cup \gamma^{-1}(q)\right) + K \le K^2 d(p, q) + K^3 + K. \tag{2.3}$$

Note that geodesics are necessarily continuous due to the local symmetry. However, the reverse of a geodesic may not be a geodesic, or even a quasigeodesic.

2.2. **BGIP and random walks.** Let us now recall the notion of bounded geodesic image property.

Definition 2.3 (Bounded geodesic image property). *A subset $A \subseteq X$ of a geodesic metric space X is said to satisfy the K-*bounded geodesic image property*, or K-BGIP in short, if the following hold:*

(1) for any $z \in X$, $\pi_A(z) \neq \emptyset$;
(2) for any geodesic η such that $\eta \cap \mathcal{N}_K(A) = \emptyset$, we have $\operatorname{diam}(\pi_A(\eta)) \le K$.

*A K-bi-quasigeodesic that satisfies K-BGIP is called a K-*BGIP axis.

One subtlety of asymmetric metrics is that contracting property and BGIP may not be equivalent anymore. In principle, we often need to replace $d(x, y)$, the distance between two points x and y, with its symmetrization $d^{sym}(x, y)$ in the arguments in [Cho22b] and [Cho22c]. For example, we now define that:

Definition 2.4 ([BF09, Definition 5.8]). *Bi-infinite paths $\kappa = (x_i)_{i \in \mathbb{Z}}$, $\eta = (y_i)_{i \in \mathbb{Z}}$ are said to be* independent *if the map $(n, m) \mapsto d^{sym}(x_n, y_m)$ is proper, i.e., for any $M > 0$, $\{(n, m) : d^{sym}(x_n, y_m) < M\}$ is bounded.*

Isometries g, h of X are said to be independent *if their orbits are independent.*

Definition 2.5. *A subgroup of* $\operatorname{Isom}(X)$ *is said to be* non-elementary *if it contains two independent contracting isometries.*

We now fix a non-elementary discrete subgroup G of $\mathrm{Isom}(X)$ and consider the random walk ω generated by a non-elementary probability measure μ on G. We employ the notions defined in [Cho22b, Subsection 2.2], except for two distinctions.

First, for a given $p > 0$, we define the p-th moment of μ by

$$\mathbb{E}_{\mu}[d(o, go)^p] := \int d(o, go)^p d\mu.$$

Note that this and the quantity

$$\mathbb{E}_{\check{\mu}}[d(o, go)^p] := \int d(o, go)^p d\check{\mu} = \int d(go, o)^p d\mu$$

are *distinct in general*, and the finitude of the former does not imply that of the latter. This technicality leads to a subtle difference between limit laws for symmetric and asymmetric metric spaces. However, many asymmetric metric spaces (including Outer space) satisfies the following coarse symmetry: there exists a global constant $K > 0$ such that $d(x, y) \le Kd(x, y)$ for $x, y \in G$. Under such a coarse symmetry, a measure μ has finite p-th moment if and only if its reflected version $\check{\mu}(\cdot) := \mu(\cdot^{-1})$ does so. Hence, the current subtlety will not matter for Outer space and many other spaces.

A measure μ is said to be *admissible* if $\langle\langle \operatorname{supp} \mu \rangle\rangle$ equals the entire group G. A measure μ is said to be *non-elementary* if $\langle\langle \operatorname{supp} \mu \rangle\rangle$ contains two independent isometries that satisfy BGIP. Note that by taking suitable powers if necessary, we may assume that two independent BGIP isometries belong to the same $\operatorname{supp} \mu^{*N}$ for some $N > 0$. μ is said to be *non-arithmetic* if there exist $N > 0$ and $g, h \in \operatorname{supp} \mu^{*N}$ such that $\tau(g) \ne \tau(h)$. Finally, we say that μ is *asymptotically asymmetric* if there exists $N > 0$ and $g, h \in \operatorname{supp} \mu^{*N}$ such that

$$\tau(g) - \tau(g^{-1}) \ne \tau(h) - \tau(h^{-1}).$$

The random walk ω generated by μ is said to be admissible (non-elementary, non-arithmetic or asymptotically asymmetric, resp.) if μ is admissible (non-elementary, non-arithmetic or asymptotically asymmetric, resp.).

2.3. **BGIP and alignment lemmata.** In [Cho22b], we collected some properties of contracting axes and deduced some alignment lemmata. We will prove the same properties for BGIP axes here.

Lemma 2.6. *Let γ be a K-bi-quasigeodesic such that $\pi_{\gamma}(y) \neq \emptyset$ for any $y \in X$. Let also $x \in \overline{\mathcal{N}_K(\gamma)}$. Then $d(x, p) \le K$ and $d(p, x) \le 3K^3 + 2K$ hold for any $p \in \pi_{\gamma}(x)$.*

Proof. Let us take $\epsilon > 0$ and $y \in \mathscr{N}_\epsilon(x) \cap \mathscr{N}_K(\gamma)$. For $p \in \pi_\gamma(x)$ and $q \in \gamma$ such that $d^{sym}(q, y) \le K$, we observe

$$\begin{aligned} d(x,p) &\le d(x,q) \le d(x,y) + d(y,q) \le \epsilon + K, \\ d(p,x) &\le d(p,q) + d(q,y) + d(y,x) \\ &\le [K^2 d(q,p) + K^3 + K] + K + \epsilon \\ &\le K^2[d(q,y) + d(y,x) + d(x,p)] + K^3 + 2K + \epsilon \\ &\le K^2[K + \epsilon + (K+\epsilon)] + K^3 + 2K + \epsilon. \end{aligned}$$

By decreasing ϵ down to zero, we deduce $d(x,p) \le K$ and $d(p,x) \le 3K^3+2K$ for any $p \in \pi_\gamma(x)$. ☐

Lemma 2.7 (Continuity of the projection). *For each $K > 1$ there exists a constant $K' = K'(K)$ that satisfies the following property.*

Let γ be a K-BGIP axis and x, y be ϵ-close points in X, that means, $d^{sym}(x,y) \le \epsilon$. Then $\pi_\gamma(\{x,y\})$ has diameter at most $K' + 2\epsilon$.

Proof. We first show $\mathrm{diam}(\pi_\gamma(w)) \le 3K^3 + 3K$ for any $w \in X$. If $w \notin \mathscr{N}_K(\gamma)$ then $\mathrm{diam}(\pi_\gamma(w)) < K$ by K-BGIP, and if $w \in \mathscr{N}_K(\gamma)$ then for any $w', w'' \in \pi_\gamma(w)$ we have $d(w', w'') \le d(w', w) + d(w, w'') \le K + (3K^3 + 2K)$ by Lemma 2.6.

Let us now prove the lemma. If one of $[x,y]$ and $[y,x]$ is disjoint from $\overline{\mathscr{N}_K(\gamma)}$, then $\mathrm{diam}(\pi_\gamma(\{x,y\})) < K$ by the BGIP. If not, we take $z \in [x,y] \cap \overline{\mathscr{N}_K(\gamma)}$ and $z' \in [y,x] \cap \overline{\mathscr{N}_K(\gamma)}$ such that $[x,z), [y,z')$ are disjoint from $\overline{\mathscr{N}_K(\gamma)}$. In other words, we take z, z' to be the 'leftmost' ones among the candidates.

Then for any $q' \in \pi_\gamma(z)$ and $q \in \pi_\gamma(y)$, we have

$$\begin{aligned} d(q',q) &\le d(q',z) + d(z,y) + d(y,q) \\ &\le 3K^3 + 2K + d(z,y) + d(y, \pi_\gamma(z')) \\ &\le 3K^3 + 2K + d(z,y) + d(y,z') + d(z', \pi_\gamma(z')) \\ &\le 3K^3 + 3K + \epsilon. \end{aligned}$$

Moreover, we have $\mathrm{diam}(\pi_\gamma([x,z])) \le 3K^3 + 3K$ since either $x = z \in \mathscr{N}_K(\gamma)$ or $[x,z]$ is disjoint from $\mathscr{N}_K(\gamma)$. Hence, we have $d(p,q) \le 6K^3 + 6K + \epsilon$ for any $p \in \pi_\gamma(x)$ and $q \in \pi_\gamma(y)$.

By symmetry, we also have $d(q,p) \le 6K^3 + 6K + \epsilon$ for such pair. Finally, we know that $\mathrm{diam}(\pi_\gamma(x)) \le 3K^3 + 3K$ and $\mathrm{diam}(\pi_\gamma(y)) \le 3K^3 + 3K$. Combining these, we conclude that $\mathrm{diam}(\pi_\gamma(x) \cup \pi_\gamma(y)) \le 12K^3 + 12K + \epsilon$. ☐

This leads to the following corollary.

Corollary 2.8 (Continuity of projections II). *Let X be a geodesic space. For each $K > 1$ there exists a constant $K' = K'(K)$ that satisfies the following property.*

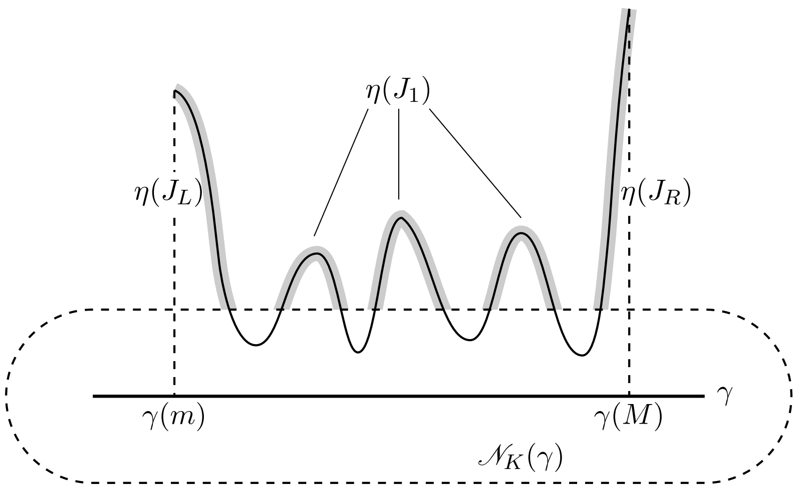

FIGURE 1. Schematics for the proof of Lemma 2.9. The projection of $\eta(J')$ onto γ is small for each component J' of $J \setminus J_0$.

Let γ be a K-BGIP axis, $A \subseteq X$ be a connected set and $a \in \mathbb{R}$. If $\gamma^{-1}\pi_\gamma(A)$ is contained in the union of $I_1 := (-\infty, a]$ and $I_2 := [a+K', +\infty)$ then it is contained in either I_1 or I_2.

Lemma 2.9 (Large projections are nearby). *For each $K > 1$ there exists a constant $K' = K'(K)$ that satisfies the following property.*

Let $\gamma : I \to X$ be a K-BGIP axis and $\eta : J \to X$ be a geodesic such that $\mathrm{diam}(\pi_\gamma(\eta)) > K$. Then for $m := \inf \gamma^{-1}\pi_\gamma(\eta)$ and $M := \sup \gamma^{-1}\pi_\gamma(\eta)$, $\gamma([m, M] \cap I)$ is within Hausdorff distance K' from a subsegment of η that contains $\mathscr{N}_K(\gamma) \cap \eta$.

Proof. Let $K_0 = K'(K)$ be as in Corollary 2.8. Let also $J_0 = \{s \in J : \eta(s) \notin \overline{\mathscr{N}_K(\gamma)}\}$, which is open since geodesics are continuous with respect to the d^{sym}-topology on X.

For each component J' of J_0, $\eta\left(\bar{J}'\right)$ is disjoint from $\mathscr{N}_K(\gamma)$ so we have $\mathrm{diam}\left(\pi_\gamma \eta\left(\bar{J}'\right)\right) \le K$. In particular, the assumption $\mathrm{diam}(\pi_\gamma(\eta)) > K$ forces that J_0 has more than 1 component; hence $J \setminus J_0$ is nonempty. We now let

$$A := \inf J \setminus J_0, \quad B := \sup J \setminus J_0$$

and claim that $\gamma([m, M] \cap I)$ and $\eta([A, B] \cap J)$ are close to each other.

First observe that each component of J_0, except the leftmost and the rightmost ones, are shorter than a uniform bound. For such a component $J' = (\alpha, \beta)$, we have $\eta(\alpha), \eta(\beta) \in \partial\mathscr{N}_K(\gamma)$ and $\mathrm{diam}\left(\pi_\gamma \eta\left([\alpha, \beta]\right)\right) < K$.

This implies that

$$\begin{aligned}|\beta-\alpha| &= d(\eta(\alpha),\eta(\beta))\\ &\le d(\eta(\alpha),\pi_\gamma\eta(\alpha)) + \operatorname{diam}(\pi_\gamma\eta(\alpha)\cup\pi_\gamma\eta(\beta)) + d(\pi_\gamma\eta(\beta),\eta(\beta))\\ &\le K + K + [3K^3+2K] =: K_1.\end{aligned}$$

Now let $s \in J$ be such that $A \le s \le B$. By its construction, s either belongs to $J \setminus J_0$ or a component $J' = (\alpha,\beta)$ of J_0 such that $\alpha,\beta \in J\setminus J_0$. In the former case, we have $d^{sym}(\eta(s),\pi_\gamma\eta(s)) \le 3K^3+3K$ by Lemma 2.6. In the latter case, for any $p \in \pi_\gamma\eta(\beta)$ we have

$$\begin{aligned}d(\eta(s),p) &\le d(\eta(s),\eta(\beta)) + d(\eta(\beta),\pi_\gamma\eta(\beta))\\ &\le K_1 + K,\\ d(p,\eta(s)) &\le \operatorname{diam}\left(\pi_\gamma(J')\right) + d(\pi_\gamma\eta(\alpha),\eta(\alpha)) + d(\eta(\alpha),\eta(s))\\ &\le K + [3K^3+2K] + K_1.\end{aligned}$$

Since $\pi_\gamma\eta(\beta) \subseteq \gamma([m,M]\cap I)$, this establishes one direction.

For the other direction, let us take $t \in I \cap [m,M]$. Let $J_L := J \cap (-\infty, A)$, $J_R := J\cap(B,+\infty)$ and $J_1 := J\cap[A,B]$. Then we have

$$\gamma^{-1}\pi_\gamma(\eta) \subseteq \gamma^{-1}\pi_\gamma(\eta(J_L)) \ \cup\ \gamma^{-1}\pi_\gamma(\eta(J_1)) \ \cup\ \gamma^{-1}\pi_\gamma(\eta(J_R)).$$

Also note that $\gamma^{-1}\pi_\gamma(\eta(J_1))$ is a K_0-connected set by Corollary 2.8, and that $\gamma^{-1}\pi_\gamma(\eta(J_L)), \gamma^{-1}\pi_\gamma(\eta(J_R))$ have diameters bounded by $2K^2$. This implies that there exists $t_0 \in I$, $s_0 \in J_1$ such that $\gamma(t_0) \in \pi_\gamma(\eta(s_0))$ and $|t-t_0| \le K_0 + 2K^2$.

If $s_0 \in J\setminus J_0$, then $d^{sym}(\gamma(t_0),\eta(s_0)) < 3K^3+3K$ by Lemma 2.6; since $\gamma(t)$ and $\gamma(t_0)$ are close to each other, we are done in this case. If $s_0 \in J_0$, it belongs to a component $J' = (\alpha,\beta)$ of J_0 that is not the leftmost or the rightmost one. We then have $\beta \in J\setminus J_0$ and $d^{sym}(\gamma(t_0),\pi_\gamma(\eta(\beta)) \le 2\operatorname{diam}\pi_\gamma(J') \le 2K$. By replacing s_0 with β and t_0 with an element of $\pi_\gamma(\eta(\beta))$, we similarly deduce the conclusion. ☐

Lemma 2.10 (BGIP is contagious)**.** *For each $K>1$ there exists a constant $K' = K'(K)$ such that any subsegment of a K-BGIP axis is a K'-BGIP axis. Moreover, if a set A is within Hausdorff distance K from a K-BGIP axis and $\pi_A(x) \neq \emptyset$ for any $x \in X$, then A has K'-BGIP.*

Proof. Let $\gamma : I \to X$ be a K-BGIP axis, $\gamma' = \gamma|_{I'} : I' \to X$ be its subsegment and $\eta : J \to X$ be a geodesic. Let also $I_L := \{x \in I : x < I'\}$, $I_R := \{x\in I : x > I'\}$. Let $K_1 = K'(K)$ be as in Lemma 2.9, $K_2 = K'(K)$ be as in Lemma 2.7, $R = 3K(K_1+K_2+K)$, $R_1 = K_1 + 2(R+1)$ and $K' = KR_1 + K^2$.

Let $z \in \eta$. We first claim that if $\gamma^{-1}(\pi_\gamma(z)) \cap I_R \neq \emptyset$, then $\gamma^{-1}\pi_{\gamma'}(z) \subseteq [\sup I' - R, \sup I']$. If not, then there exists $w \in \pi_{\gamma'}(z)$ such that $\gamma^{-1}(w)$ intersects $(-\infty, \sup I' - R)$. Then $\operatorname{diam}(\pi_\gamma(z)\cup w) \ge R/K - K > K+1$ so $[z,w]$ passes through $\mathscr{N}_{K_1}(\gamma(\sup I'))$ by Lemma 2.9. Let $p \in [z,w]$ be that intersection point. Then

$$\begin{aligned}
d(z, \gamma(\sup I' - \epsilon)) &\le d(z, \gamma(\sup I')) + K\epsilon + K \\
&\le d(z,p) + d(p, \gamma(\sup I')) + K\epsilon + K \\
&\le d(z,w) - d(p,w) + d(p, \gamma(\sup I')) + K\epsilon + K \\
&\le d(z,w) - d(\gamma(\sup I'), w) + d(\gamma(\sup I'), p) + d(p, \gamma(\sup I')) + K\epsilon + K \\
&\le d(z,w) + K_1 + K\epsilon + K - (R/K - K) < d(z,w)
\end{aligned}$$

for sufficiently small $\epsilon > 0$, which is a contradiction.

By a similar reason, $\gamma^{-1}(\pi_\gamma(z)) \cap I_L \neq \emptyset$ implies $\pi_{\gamma'}(z) \subseteq \gamma([\inf I', \inf I' + R])$. Finally, if $\gamma^{-1}(\pi_\gamma(z)) \cap I' \neq \emptyset$ then $\pi_{\gamma'}(z) = \pi_\gamma(z) \cap \gamma'$.

Let us now suppose that the diameter of $\pi_{\gamma'}(\eta)$ is greater than K'. Without loss of generality, let $x, y \in \eta$ and $s' \in \gamma^{-1}\pi_{\gamma'}(x)$, $t' \in \gamma^{-1}\pi_{\gamma'}(y)$ be such that $t' - s' > K'/K - K = R_1$. We then pick s to be s' if $s' \in \gamma^{-1}\pi_\gamma(x)$ and an arbitrary element of $\gamma^{-1}\pi_\gamma(x)$ if not. Similarly we take $t = t'$ or an element of $\gamma^{-1}\pi_\gamma(y)$.

We claim that $s \le s' + R + 1$. If not, we have either $s \in I_R$ or $s' \le s - R - 1 \le \sup I' - R - 1$. In the former case we have $\sup I' - R \le s' < t' \le \sup I'$ and $t' - s' \le R < R_1$, a contradiction. In the latter case, the previous observation tells us that $\gamma^{-1}(\pi_\gamma(x)) \cap I_R = \emptyset$. This forces one of the following cases:

- $\gamma^{-1}(\pi_\gamma(x)) \cap I' \neq \emptyset$ holds, in which case $\pi_{\gamma'}(x) = \pi_\gamma \cap \gamma'(x)$ and $|s - s'| \le K \operatorname{diam}(\pi_\gamma(x)) + K^2 \le R$ hold; or,
- $\gamma^{-1}(\pi_\gamma(x)) < I'$ and $s \le s'$; in either case we have a contradiction.

By a similar reason, we also deduce $t \ge t' - R - 1$. In conclusion, we have

$$t - s \ge \min(t, t') - \max(s, s') \ge t' - s' - 2(R+1) \ge R_1 - 2(R+1) \ge K_1 \tag{2.4}$$

and $\mathscr{N}_K(\gamma(s^*)) \cap \eta \neq \emptyset$ for all $s \le s^* \le t$ by K-BGIP of γ. Also, Inequality 2.4 implies that $s^* \in [\min(t, t'), \max(s, s')]$ exists, which clearly belongs to I'. This establishes K'-BGIP of γ'.

We now investigate the second assertion. Let

$$K_3 := 2K^2(10K^3 + K_1 + K_2) + K_2.$$

As before, let $\gamma : I \to X$ be a K-BGIP axis and A be a K-bi-quasigeodesic that is within Hausdorff distance K from γ. For $x \in X$, we claim that $\pi_\gamma(x) \cup \pi_A(x)$ is bounded. To see this, let $z \in \pi_\gamma(x)$ and $z' \in \pi_A(x)$. Since γ and A are within Hausdorff distance K, there exist $w \in \gamma$, $w' \in A$ such that $d^{sym}(w, z'), d^{sym}(w', z) \le K$. Then for any $w^* \in \pi_\gamma(z')$ we have

$$\begin{aligned}
\operatorname{diam}(z' \cup w^*) &\le d(z', w^*) + d(w^*, z') \\
&\le d(z', w) + d(w^*, w) + d(w, z') \\
&\le d^{sym}(w, z') + K^2 d(w, w^*) + K^3 + K \\
&\le d^{sym}(w, z') + K^2[d(w, z') + d(z', w^*)] + K^3 + K \\
&\le (K^2 + 1) d^{sym}(w, z') + K^3 + K \le 2K^3 + 2K.
\end{aligned}$$

Now, if $d(z,z') \geq 2K^3 + 3K + K_1 + K_2$, then

$$\mathrm{diam}(\pi_\gamma([x,z'])) \geq \mathrm{diam}(z \cup \pi_\gamma(z')) \geq \mathrm{diam}(z \cup z') - \mathrm{diam}(\pi_\gamma(z') \cup z) \geq K$$

and $[x,z']$ passes through $\mathscr{N}_{K_1}(z)$ by K-BGIP of γ. Let $p \in [x,z']$ be a point in the intersection. This implies that

$$\begin{aligned} d(x,w') &\leq d(x,p) + d(p,w') \\ &\leq d(x,z') - d(p,z') + d(p,z) + d(z,w') \\ &\leq d(x,z') - [d(z,z') - d(z,p)] + d(p,z) + d(z,w') \\ &\leq d(x,z') - (2K^3 + 3K + K_1 + K_2) + K_1 + K < d(x,z'), \end{aligned}$$

which contradicts the fact that $z' \in \pi_A(x)$. Hence, we conclude that $d(z,z') < 2K^3 + 3K + K_1 + K_2$ and $d(z,w) \leq 2K^3 + 4K + K_1 + K_2$. Since γ is a K-bi-quasigeodesic, we have

$$\begin{aligned} d(w,z) &\leq K^2(2K^3 + 4K + K_1 + K_2) + K^3 + K, \\ d(z',z) &\leq K^2(2K^3 + 4K + K_1 + K_2) + K^3 + 2K \\ &\leq K^2(10K^3 + K_1 + K_2). \end{aligned}$$

In short, we have $d^{sym}(z,z') \leq K_3 - K_2$. Since $\mathrm{diam}(\pi_\gamma(x)) \leq K_2$ by Lemma 2.7, we conclude that $\mathrm{diam}(\pi_\gamma(x) \cup \pi_A(x)) \leq K_3$.

Now suppose $\mathrm{diam}(\pi_A([x,y])) > 2K_3 + K$. By the previous argument, we deduce that $\mathrm{diam}(\pi_\gamma([x,y])) > K$ and $[x,y]$ passes through $\mathscr{N}_K(\gamma) \subseteq \mathscr{N}_{2K}(A)$. Hence, A has $(2K_3 + 2K)$-BGIP. □

Lemma 2.11 (No backtracking). *For each $K > 1$ there exists a constant $K' = K'(K)$ that satisfies the following property.*

Let $\gamma : I \to X$ be a K-BGIP axis, $\eta : J \to X$ be a geodesic and $\alpha_i \in J$ be such that $\alpha_1 \leq \alpha_2 \leq \alpha_3$. Let also $a_1, a_2, a_3 \in I$ be such that $\gamma(a_i) \in \pi_\gamma \eta(\alpha_i)$. Then a_1 and a_3 cannot both belong to $(-\infty, a_2 - K']$ nor $[a_2 + K', +\infty)$.

Proof. Let $K_1 = K'(K)$ be as in Lemma 2.9 and $K_2 = K'(K)$ be as in Lemma 2.7. We claim that $K' = K(2K + 3K_1 + 1)$ works.

Suppose first that $a_1, a_3 \in [a_2 + K', +\infty)$. Let $a := \min\{a_1, a_3\}$. We then have $a \in [a_2, a_1]$, $\gamma(a_i) \in \pi_\gamma \eta(\alpha_i)$ and

(2.5)
$$\mathrm{diam}(\pi_\gamma \eta([\alpha_1, \alpha_2])) \geq \mathrm{diam}(\pi_\gamma \eta(\alpha_2) \cup \pi_\gamma \eta(\alpha_1)) > \frac{1}{K}|a_1 - a_2| - K > K + 1.$$

Hence, by Lemma 2.9, there exists $w_1, w_2 \in [\alpha_1, \alpha_2]$ such that

$$d^{sym}(\eta(w_1), \gamma(a)) < K_1, \quad d^{sym}(\eta(w_2), \gamma(a_2)) < K_1.$$

Similarly, we have $w_1', w_2' \in [\alpha_2, \alpha_3]$ such that

$$d^{sym}(\eta(w_1'), \gamma(a)) < K_1, \quad d^{sym}(\eta(w_2'), \gamma(a_2)) < K_1.$$

Meanwhile, Inequality 2.5 also shows that $\mathrm{diam}(\pi_\gamma([\eta(\alpha_1), \gamma(a_2)]))$ is larger than $K+1$. Since $a_2 \leq a \leq a_1$, Lemma 2.9 implies that $[\eta(\alpha_1), \gamma(a_2)]$ passes

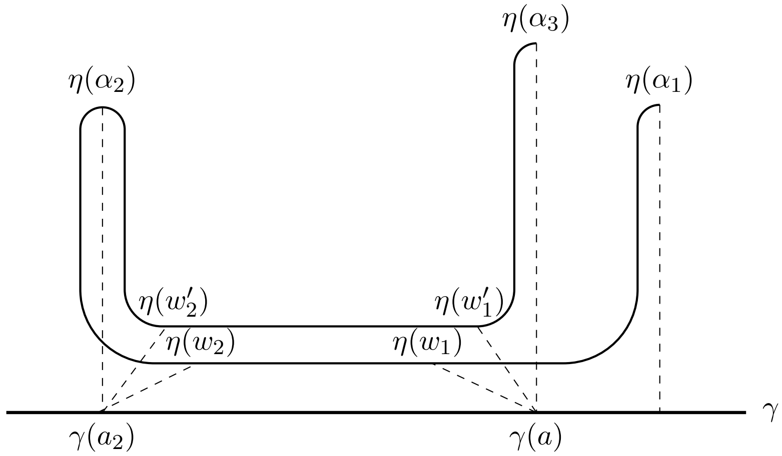

FIGURE 2. Schematics for the proof of Lemma 2.11.

through $\mathscr{N}_{K_1}(\gamma(a))$. Let p be the intersection point and note that

$$\begin{aligned}
d(\eta(\alpha_1), \eta(\alpha_2)) &\geq d(\eta(\alpha_1), \eta(w_2)) \\
&\geq d(\eta(\alpha_1), \gamma(a_2)) - d(\eta(w_2), \gamma(a_2)) \\
&= d(\eta(\alpha_1), p) + d(p, \gamma(a_2)) - d(\eta(w_2), \gamma(a_2)) \\
&\geq [d(\eta(\alpha_1), \gamma(a)) - d(p, \gamma(a))] + [d(\gamma(a), \gamma(a_2)) - d(\gamma(a), p)] - 2K_1 \\
&\geq d(\eta(\alpha_1), \gamma(a)) + \left[\frac{1}{K}|a - a_2| - K\right] - d^{sym}(p, \gamma(a)) - 2K_1 \\
&\geq d(\eta(\alpha_1), \gamma(a)) + \frac{K'}{K} - K - 3K_1.
\end{aligned}$$

By a similar reason, $[\gamma(a_2), \eta(\alpha_3)]$ passes through $\mathscr{N}_{K_1}(\gamma(a))$ and we can deduce

$$d(\eta(\alpha_2), \eta(\alpha_3)) \geq d(\gamma(a), \eta(\alpha_3)) + \frac{K'}{K} - K - 3K_1.$$

Since $\eta(\alpha_1)$, $\eta(\alpha_2)$ and $\eta(\alpha_3)$ are aligned on the same geodesic η, we deduce

$$\begin{aligned}
d(\eta(\alpha_1), \eta(\alpha_3)) &= d(\eta(\alpha_1), \eta(\alpha_2)) + d(\eta(\alpha_2), \eta(\alpha_3)) \\
&\geq d(\eta(\alpha_1), \gamma(a)) + d(\gamma(a), \eta(\alpha_3)) + 2\left(\frac{K'}{K} - K - 3K_1\right) \\
&\geq d(\eta(\alpha_1), \eta(\alpha_3)) + 2\left(\frac{K'}{K} - K - 3K_1\right).
\end{aligned}$$

Since $K' > K(K + 3K_1)$, this gives a contradiction. Similar investigation also prevents $a_1, a_3 \in (-\infty, a_2 - K']$. □

Lemma 2.12 (Fellow traveling)**.** *For each $K > 1$ there exists a constant $K' = K'(K)$ that satisfies the following property.*

Let $\gamma : I \to X$ be a K-bi-quasigeodesic and $\eta_1 : [0, L_1] \to X$, $\eta_2 : [0, L_2] \to X$ be geodesics such that $d_H(\gamma, \eta_1), d_H(\gamma, \eta_2) < K$. Suppose also *that* $d(\eta_1(0), \eta_2(0)) < K$*. Then* $|L_1 - L_2| < K'$*, and* η_1 *and* η_2 K'*-fellow travel on the interval* $[0, \min\{L_1, L_2\}]$*.*

Proof. For each $0 \le s_1 \le L_1$, let $t \in I$ be such that $d^{sym}(\eta_1(s_1), \gamma(t)) < K$ and let $s_2 \in [0, L_2]$ be such that $d^{sym}(\eta_2(s_2), \gamma(t)) < K$. Then we have

$$\begin{aligned}|s_1 - s_2| &= |d(\eta_1(0), \eta_1(s_1)) - d(\eta_2(0), \eta_2(s_2))| \\ &\le d^{sym}(\eta_1(0), \eta_2(0)) + |d(\eta_2(0), \eta_2(s_2)) - d(\eta_2(0), \eta_2(s_2))| + d^{sym}(\eta_2(s_2), \eta_1(s_1)) \\ &\le 4K.\end{aligned}$$

In particular, this implies that $L_2 \ge L_1 - 4M$. By symmetry, $L_1 \ge L_2 - 4M$ also holds.

Now for $0 \le s_1 \le \min\{L_1, L_2\}$, define t and s_2 as above. Let also $t' \in I$ be such that $d^{sym}(\eta_2(s_1), \gamma(t')) < K$. We then have

$$\begin{aligned}d(\eta_1(s_1), \eta_2(s_1)) &\le d(\eta_1(s_1), \eta_2(s_2)) + d(\eta_2(s_2), \eta_2(s_1)) \\ &\le 2K + d(\eta_2(s_2), \gamma(t)) + d(\gamma(t), \gamma(t')) + d(\gamma(t'), \eta_2(s_1)) \\ &\le 2K + 2K + K|t - t'| + K \\ &\le 5K + K^2 d(\gamma(t'), \gamma(t)) + K^2 \\ &\le (5K + K^2) + K^2[d(\gamma(t'), \eta_2(s_1)) + d(\eta_2(s_1), \eta_2(s_2)) + d(\eta_2(s_2), \gamma(t))] \\ &\le (5K + K^2 + 2K^3) + K^2 d(\eta_2(s_1), \eta_2(s_2)).\end{aligned}$$

Since one of $d(\eta_2(s_2), \eta_2(s_1))$ and $d(\eta_2(s_1), \eta_2(s_2))$ is bounded by $|s_1 - s_2| \le 4K$, we conclude that $d(\eta_1(s_1), \eta_2(s_1)) \le 6K + K^2 + 6K^3$. Similar estimate holds for $d(\eta_2(s_1), \eta_1(s_1))$. □

Thanks to these lemmata, we can copy the alignment lemmata in [Cho22b] by replacing K-contracting axes with K-BGIP axes.

Definition 2.13 ([Cho22b, Definition 3.2])**.** *Given paths* κ_i *from* x_i *to* x'_i *for each* $i = 1, \ldots, n$*, we say that* $(\kappa_1, \ldots, \kappa_n)$ *is* C*-aligned if*

$$\operatorname{diam}\left(x'_i \cup \pi_{\kappa_i}(\kappa_{i+1})\right) < C, \quad \operatorname{diam}\left(x_{i+1} \cup \pi_{\kappa_{i+1}}(\kappa_i)\right) < C.$$

hold for $i = 1, \ldots, n-1$*.*

Lemma 2.14 ([Cho22b, Lemma 3.3])**.** *For each* $C > 0$ *and* $K > 1$*, there exists* $D = D(K, C) > C$ *that satisfies the following property.*

Let κ, η *be* K*-BGIP axes that connect* x *to* x' *and* y *to* y'*, respectively. Suppose that* (κ, y') *and* (x, η) *are* C*-aligned. Then* (κ, η) *is* D*-aligned.*

Proposition 2.15 ([Cho22b, Proposition 3.5])**.** *For each* $C > 0$ *and* $K > 1$*, there exist* $D = D(K, C) > C$ *and* $L = L(K, C) > C$ *that satisfies the following.*

Let J *be a nonempty set of consecutive integers, and* $p, \{x_i, y_i\}_{i \in J}$ *be points in* X*. For each* $i \in J$*, let* κ_i *be a* K*-BGIP axis connecting* x_i *to* y_i *whose domain is longer than* L*. Suppose also that* $(\kappa_i)_{i \in J}$ *is* C*-aligned. Then we have the following:*

(1) the statements

$$(\kappa_i, p) \text{ is } D\text{-aligned}, \quad (p, \kappa_i) \text{ is } D\text{-aligned}$$

cannot hold simultaneously;

(2) the set

$$J_0 = J_0\Big(p; (\kappa_i)_{i \in J}, D\Big) := \left\{ j \in J : \begin{array}{l} (\kappa_i, p) \text{ is } D\text{-aligned for } i \in J \text{ such that } i < j, \\ (p, \kappa_i) \text{ is } D\text{-aligned for } i \in J \text{ such that } i > j \end{array} \right\}$$

consists of either a single integer or two consecutive integers;

(3) $\pi_{\cup_i \kappa_i}(p)$ *is nonempty and is contained in* $\bigcup\{\pi_{\kappa_j}(p) : j \in J_0\}$*; and*

(4) (κ_l, κ_m) *is* D*-aligned for any* $l, m \in J$ *such that* $l < m$*.*

Proposition 2.16 ([Cho22b, Proposition 3.6])**.** *For each* $C > 0$ *and* $K > 1$*, there exist* $E = E(K, C) > C$ *and* $L = L(K, C) > C$ *that satisfy the following. Let* $x, y \in X$ *and* $\kappa_1, \ldots, \kappa_N$ *be* K*-BGIP axes whose domains are longer than* L*.*

If $(x, \kappa_1, \ldots, \kappa_N, y)$ *is* C*-aligned, then* (x, κ_i, y) *is* E*-witnessed for each* $i = 1, \ldots, N$*. Moreover,* $p \in \mathcal{N}_E([x, y])$ *and* $(x, y)_p < E$ *for any* $p \in \kappa_i$*.*

Lemma 2.17 ([Cho22b, Lemma 3.7])**.** *For each* $C, M > 0$ *and* $K > 1$*, there exist* $K' = K'(K, C, M) > C$ *and* $L = L(K, C) > C$ *that satisfies the following.*

Let J *be a nonempty set of consecutive integers and* $\{x_i, y_i\}_{i \in J}$ *be points in* X*. For each* $i \in J$*, let* κ_i *be a* K*-BGIP axis connecting* x_i *and* y_i *whose domain is longer than* L*. Suppose that* $(\kappa_i)_{i \in J}$ *is* C*-aligned and* $d(y_i, x_{i+1}) < M$ *for* $i \in J \setminus \sup J$*. Then* $\cup_i \kappa_i$ *is a* K'*-BGIP axis.*

We now recall the concept of Schottky sets. Given a sequence $s = (\phi_i)_{i=1}^k$ of isometries of X, we denote the product of its entries $\phi_1 \cdots \phi_k$ by $\Pi(s)$. We also define the reversal of s by $s^{-1} := (\phi_{k-i+1}^{-1})_{i=1}^k$, i.e.,

$$s = (\phi_1, \ldots, \phi_k) \;\Leftrightarrow\; s^{-1} = (\phi_k^{-1}, \ldots, \phi_1^{-1}).$$

Now let

$$x_{nk+i} := \Pi(s)^n \phi_1 \cdots \phi_i o = (\phi_1 \cdots \phi_k)^n \phi_1 \cdots \phi_i o$$

for each $n \in \mathbb{Z}$ and $i = 0, \ldots, k-1$. We let $\Gamma^m(s) := (x_0, x_1, \ldots, x_{mk})$ when $m \geq 0$ and $\Gamma^m(s) := (x_0, x_{-1}, \ldots, x_{mk})$ when $m < 0$. When $m = 1$, we usually omit the superscript and write $\Gamma(s) = (x_0, \ldots, x_k)$. Finally, let $\Gamma^{\pm\infty}(s) = (x_i)_{i \in \mathbb{Z}}$. Note that $\Gamma^{-m}(s) = \Gamma^m(s^{-1})$, and $\Gamma^m(s)$ is a concatenation of $|m|$ translates of $\Gamma(s)$ or its reverse.

Definition 2.18 ([Cho22b, Definition 3.11])**.** *Let* $K > 0$ *and* $S \subseteq G^M$ *be a set of sequences of* M *isometries. We say that* S *is* K-Schottky *if the following hold:*

(1) $\Gamma^m(s)$ *is a* K*-BGIP axis for all* $s \in S$ *and* $m \in \mathbb{Z}$*;*

(2) for each $x \in X$, all element $s \in S$ except at most 1 satisfies that $(x, \Gamma^n(s))$ is K-aligned for all $n \in \mathbb{Z}$;

(3) for each $x \in X$ and $s \in S$, if $(x, \Gamma^n(s))$ is not K-aligned for some $n > 0$ ($n < 0$, resp.) then $(x, \Gamma^m(s))$ is K-aligned for all $m \le 0$ ($m \ge 0$, resp.).

Proposition 2.19 ([Cho22b, Proposition 3.12])**.** *For any $N_0 > 0$, there exists a K-Schottky set of cardinality N_0 in $(\operatorname{supp} \mu)^m$ for some m and K.*

From now on we fix an integer $N_0 > 410$. Let $K_0 := K(N_0)$ be as in Proposition 2.19, and

- $D_0 := D(K_0, K_0)$ be as in Lemma 2.14;
- for $i = 1, 2$, $D_i := D(K_0, D_{i-1})$, $L_i := L(K_0, D_{i-1})$ be as in Lemma 2.14 and Proposition 2.15;
- $E_0 := E(K_0, D_2)$, $L_3 := L(K_0, D_2)$ be as in Proposition 2.16.

Let us now fix a K_0-Schottky set $S \subseteq (\operatorname{supp} \mu)^{M_0}$ of cardinality at least N_0.

Note that the set of n-self-concatenations of elements of S is also a K_0-Schottky set. Hence, we may assume that

$$M_0 > L_1 + L_2 + L_3 + 20K_0(K_0 + E_0). \tag{2.6}$$

From now on, K_0-BGIP axes of the form $\Gamma^m(s)$ for $s \in S$ and $m \neq 0$ are called *Schottky axes.*

Our choice of constants is important: since M_0 is larger than L_3, Proposition 2.15 tells us the following.

Corollary 2.20. *Let $(\Gamma_1, \ldots, \Gamma_N)$ be a D_2-aligned sequence of Schottky axes. Let us also define a relation for $x, y \in \cup_i \Gamma_i$: $x \prec y$ if*

- *$x \in \gamma_i$ and $y \in \gamma_j$ for some $i < j$, or*
- *x and y are the beginning and terminating points, respectively, of the same axis Γ_j for some j.*

Then $y \in \mathcal{N}_{E_0}([x, z])$ and $(x, z)_y < E_0$ hold for any triple $x, y, z \in \cup_i \gamma_i$ such that $x \prec y \prec z$.

3. Outer space

The aim of this section is to collect facts regarding the outer automorphism group and Outer space. For detailed definitions and theories, see the general exposition of Vogtmann [Vog15] or individual papers, e.g. [BH92], [FM11], [FM12], [AKB12], [AK11], [DH18], [DT18] and [KMPT22].

Let X be the Culler-Vogtmann Outer space CV_N of rank $N \ge 3$, which is the space of unit-volume marked metric graphs with fundamental group F_N. In other words, a point $p \in CV_N$ corresponds to the homotopic class of a homotopy equivalence $h : R_N \to \Gamma$, where R_N is a fixed rose with N petals and Γ is a unit-volume metric graph. The corresponding space without the volume normalization is called the unprojectivized Outer space cv_N, and there is a projectivization from cv_N to CV_N by dilation.

Outer space comes equipped with a canonical metric, the Lipschitz distance, which is defined as follows: for two markings $h_1 : R_N \to \Gamma_1$ and $h_2 : R_N \to \Gamma_2$, the distance from Γ_1 to Γ_2 is defined by

$$d_{CV}(\Gamma_1, \Gamma_2) := \inf\{\log \mathrm{Lip}(f) : f \sim f_2 \circ f_1^{-1}\},$$

where $\mathrm{Lip}(f)$ is the (maximal) Lipschitz constant of f. *We now make a convention that differs from the traditional one.* Namely, the outer automorphism group $\mathrm{Out}(F_N)$ of rank N acts on CV_N by changing the basis of the marking *with the inverses*: given $\phi \in \mathrm{Out}(F_N)$ and $h : R_N \to \Gamma$ representing a point of CV_N, ϕ moves h to $h \circ \phi^{-1} : F_N \stackrel{\phi^{-1}}{\to} F_N \stackrel{h}{\to} \Gamma$. This is a left action by isometries. We denote action by $X \ni h \mapsto \phi \cdot h \in X$.

It is known that the Lipschitz distance is asymmetric [FM11] and not uniquely geodesic. However, distances among ϵ-thick points (i.e., those with systole at least ϵ) have the coarse symmetry: there exists a constant $C = C(\epsilon) < +\infty$ such that for any ϵ-thick points x and y, one has $d(x, y) \le Cd(y, x)$ [AKB12]. In particular, distances among the translates of the reference point o by $\mathrm{Out}(F_N)$ satisfy the coarse symmetry.

Just as Teichmüller space $\mathcal{T}(\Sigma)$ is accompanied by the curve complex $\mathcal{C}(\Sigma)$ and the coarse projection $\pi^{\mathcal{C}} : \mathcal{T}(\Sigma) \to \mathcal{C}(\Sigma)$, CV_N is accompanied by the complex of free factors $\mathcal{FF}_N$ and the coarse projection $\pi^{\mathcal{FF}} : CV_N \to \mathcal{FF}_N$. This projection is coarsely $\mathrm{Out}(F_N)$-equivariant and coarsely Lipschitz. Moreover, geodesics in CV_N projects to K-unparametrized biquasigeodesics for some uniform $K > 0$ [BF14, Proposition 9.2].

Outer space also accomodates lots of BGIP isometries. We say that an outer automorphism $\phi \in \mathrm{Out}(F_N)$ is *reducible* if there exists a free product decomposition $F_N = C_1 * \cdots * C_k * C_{k+1}$, with $k \ge 1$ and $C_i \neq \{e\}$, such that ϕ permutes the conjugacy classes of $C_1, \ldots, C_k$. If not, we say that ϕ is *irreducible*. We also say that ϕ is *fully ireducible (or iwip)* if no power of ϕ is reducible, or equivalently, no power of ϕ preserves the conjugacy class of any proper free factor of F_N. We also say that ϕ is *atoroidal (or hyperbolic)* if no power of ϕ fixes any nontrivial conjugacy class in F_N. When ϕ is fully irreducible, it is non-atoroidal if and only if it is *geometric*, i.e., induced by a pseudo-Anosov $\varphi : \Sigma \to \Sigma$ on a compact surface Σ with one boundary component, via an identification of F_N with $\pi(\Sigma)$. Bestvina and Feighn proved in [BF14] that $\phi \in \mathrm{Out}(F_N)$ is fully irreducible if and only if it acts on $\mathcal{FF}_N$ loxodromically.

We say that a subgroup $G \le Out(F_N)$ is *non-elementary* if it acts on $\mathcal{FF}_N$ in a non-elementary way, or equivalently, contains two fully irreducibles with mutually distinct attracting/repelling trees. It is known that if $G \le Out(F_N)$ does not fix any finite subset of $\mathcal{FF}_N \cup \partial\mathcal{FF}_N$, or equivalently, if it is not virtually cyclic nor virtually fixes the conjugacy class of a proper free factor of F_N, then G is non-elementary [Hor16b]. Since $\pi^{\mathcal{FF}}$ is coarsely

Lipschitz, the independence of two fully irreducibles in $\mathcal{FF}_N$ is lifted to the independence in CV_N.

We refer the readers to [BH92], [AK10], [BF14] and [AKKP19] for the precise definition of a train-track representative $f : \Gamma \to \Gamma$ of an outer automorphism ϕ. Roughly speaking, a train-track representative of ϕ is a self-map $f : \Gamma \to \Gamma$ in the free homotopy class of ϕ on a simplicial graph Γ that sends vertices to vertices, restricts to an immersion on each edge of Γ, and sends edges to immersed segments after iterations. It is due to Bestvina and Handel [BH92] that every irreducible outer automorphism admits a train-track representative, although it may not be unique.

Given such a structure, one can endow Γ with a metric such that f stretches each edge of Γ by the same constant $\lambda > 1$, which is called the *expansion factor* of f. This expansion factor is uniquely determined by the choice of ϕ and does not depend on the choice of f. Moreover, in view of Skora's interpretation of Stallings fold decompositions, one obtains a continuous path on cv_N from Γ to $\Gamma \circ \phi$ by folding a single illegal turn at each time (cf. [AKKP19]). This descends to a geodesic segment of length $\log \lambda$ (after a reparametrization) and the concatenation of its translates by powers of ϕ becomes a bi-infinite, ϕ-periodic geodesic. We call this a *(optimal) folding axis* of ϕ. Algom-Kfir observed the following:

Theorem 3.1 ([AK11])**.** *Folding axes of fully irreducible outer automorphisms are strongly contracting.*

Unfortunately, we need BGIP instead of the strongly contracting property in our setting, and the author does not know a way to promote the latter to the former. Meanwhile, I. Kapovich, Maher, Pfaff and Taylor observed the following version of BGIP in Outer space. This requires the notion of greedy folding path, whose accurate definition can be found in [FM11], [BF14] and [DH18]. In short, a greedy folding path $\gamma : I \to cv_N$ is obtained by folding every illegal turns at each time with speed 1, where the illegal turn structures at different forward times are identical and define a well-defined illegal turn structure. This also descends to a geodesic on CV_N, and we have the following theorem:

Theorem 3.2 ([KMPT22, Theorem 7.8])**.** *Let $\phi \in \mathrm{Out}(F_N)$ be a fully irreducible outer automorphism. Suppose that γ is a bi-infinite, ϕ-periodic greedy folding path. Then there exist $C > 0$ such that the following holds.*

Let $x, y \in X$ be points such that $d^{sym}(\pi_\gamma(x), \pi_\gamma(y)) \geq C$, and satisfy $d^{sym}(\pi_\gamma(x)) = \gamma(t_1)$, $d^{sym}(\pi_\gamma(y)) = \gamma(t_2)$ for some $t_1 < t_2$. Then any geodesic $[x, y]$ between them contains a subsegment $[z_1, z_2]$ such that

$$d^{sym}(z_1, \pi_\gamma(x)) < C, \quad d^{sym}(z_2, \pi_\gamma(y)) < C.$$

This uni-directional version of BGIP is designed for outer automorphisms that have an invariant greedy folding line. It seems not shown that all fully irreducibles have such a line. (The author thanks Sam Taylor for pointing this out.) Nonetheless, by adapting Dowdall-Taylor's idea and Kapovich-Maher-Pfaff-Taylor's proof of Theorem 3.2, we can obtain the following result. This proof was kindly informed by Sam Taylor.

Proposition 1.3. *Let $\varphi \in \mathrm{Out}(F_N)$ be a fully irreducible outer automorphism. Then the orbit $\{\varphi^i o\}_{i\in\mathbb{Z}}$ of o by φ is a BGIP axis.*

Proof. Before we begin, we recall the following facts regarding a geodesic δ-hyperbolic space Y.

(1) (Morse property) A K-quasigeodesic and a geodesic with the same endpoints are within Hausdorff distance $K_2 = K_2(K, \delta)$.
(2) The closest point projections onto a K-quasigeodesic and a geodesic on Y with the same endpoints are within distance $K_3 = K_3(K, \delta)$.
(3) If the projections of $x, y \in Y$ to K-quasigeodesic γ contain $\gamma(s)$ and $\gamma(t)$, respectively, and $d(\gamma(s), \gamma(t)) > K_4 = K_4(K, \delta)$, then $[x, y]$ and $[x, \pi_\gamma(x)] \cup \gamma|_{[s,t]} \cup [\pi_\gamma(y), y]$ are within Hausdoff distance K_4.
(4) If K-quasigeodesics γ, γ' are within Hausdorff distant K and the distance between starting points is at most K, then γ' crosses γ up to a constant $K_5 = K_5(K, \delta)$, i.e., γ and $\gamma' \circ \rho$ K_5-fellow travel for some orientation-matching reparametrization ρ.

Let T^+, T^- be the attracting and repelling trees of φ, respectively. There exist optimal greedy folding lines $\gamma^\pm : \mathbb{R} \to CV_N$ such that

$$\lim_{t\to+\infty} \gamma^\pm(t) = T^\pm, \quad \lim_{t\to-\infty} \gamma^\pm(t) = T^\mp \tag{3.1}$$

([BR15], Lemma 6.7 and Lemma 7.3). Since $\{\varphi_i o\}_i$ is a quasigeodesic whose endpoints agree with γ^+, Theorem 4.1 of [DT18] asserts that $d_H(\{\varphi^i o\}_i, \gamma^+) < K_1$ and $\pi^{\mathcal{FF}}(\gamma^+)$ is a K_1-quasigeodesic for some K_1. Similarly, by comparing $\{\varphi_{-i} o\}_i$ and γ^-, we deduce that $d_H(\{\varphi^i o\}_i, \gamma^-) < K_1$ and $\pi^{\mathcal{FF}}(\gamma^-)$ is a K_1-quasigeodesic. Also, $\gamma^\pm$ are uniformly thick.

Let us now take $x_i^+ \in \pi_{\gamma^+}(\varphi^i o)$ and $x_i^- \in \pi_{\gamma^-}(\varphi^i o)$ for each i. We recall the following result of Dahmani and Horbez ([DH18, Proposition 5.17, Corollary 5.22]; see also Section 7 of [KMPT22]): there exist $B, D > 0$ such that $\gamma^\pm$ are (B, D)-contracting at $x_i^\pm$'s (with a suitable crossing constant κ). In other words, a geodesic η on CV_N projects to a path that κ-crosses up a large enough subsegment of $\pi^{\mathcal{FF}}\gamma^\pm$ that begins from $\pi^{\mathcal{FF}}(x_i^\pm)$, then η has a point p whose distance to $\gamma^\pm$ is bounded by D. Since $\gamma^\pm$ are thick, the distance from $\gamma^\pm$ to such point p is also controlled and η intersects a neighborhood of $\gamma^\pm$ in such a case.

We now observe that $\pi^{\mathcal{FF}}\pi_{\gamma^+}, \pi^{\mathcal{FF}}\pi_{\gamma^-}$ and $\pi_{\pi^{\mathcal{FF}}(\{\varphi^i o\}_i)} \circ \pi$ are coarsely equivalent. First, Lemma 4.11 of [DT18] asserts that $\pi_{\gamma^\pm}$ and $Pr_{\gamma^\pm}$ are equivalent, where Pr stands for the Bestvina-Feighn left projection. Then Lemma 4.2 of the same paper asserts that $\pi^{\mathcal{FF}} Pr_{\gamma^\pm}$ and $\pi_{\pi^{\mathcal{FF}}(\gamma^\pm)} \circ \pi$ are equivalent. These are then equivalent to $\pi_{\pi^{\mathcal{FF}}(\{\varphi^i o\}_i)} \circ \pi$, since $\pi^{\mathcal{FF}}(\gamma^\pm)$ and $\pi^{\mathcal{FF}}(\{\varphi^i o\}_i)$ are close to each other and $\pi^{\mathcal{FF}}(\{\varphi^i o\}_i)$, a quasi-geodesic on the Gromov hyperbolic space $\mathcal{FF}$, is strongly contracting.

We now lift these projections: we claim that π_{γ^+}, π_{γ^-} and $\pi_{\{\varphi^i o\}_i}$ are equivalent. First, suppose that $\pi_{\gamma^+}(x)$ and $\pi_{\gamma^-}(x)$ are far from each other for some $x \in X$. Since γ^+, γ^-, $\{\varphi^i o\}_i$ are close to each other, we may take $\varphi^i o$ and $\varphi^j o$ near $\pi_{\gamma^+}(x)$ and $\pi_{\gamma^-}(x)$, respectively, and conclude that $|i-j|$ is large. This implies that $\pi^{\mathcal{FF}}(\varphi^i o)$ and $\pi^{\mathcal{FF}}(\varphi^j o)$ are also far from each other (since φ is loxodromic on CV_N), and consequently $\pi^{\mathcal{FF}}(\pi_{\gamma^+}(x))$, $\pi^{\mathcal{FF}}(\pi_{\gamma^-}(x))$ are far from each other. $(*)$ Since we have proved that $\pi^{\mathcal{FF}}\pi_{\gamma^+}$ and $\pi^{\mathcal{FF}}\pi_{\gamma^-}$ are equivalent, this cannot happen. Hence, π_{γ^+} and π_{γ^-} are equivalent.

Now suppose that $\pi_{\{\varphi^i o\}_i}(x)$ and $\pi_{\gamma^\pm}(x)$ are far from each other for some $x \in X$. We take $\varphi^j o \in \pi_{\{\varphi^i o\}_i}(x)$ and $\varphi^{j'} o$ near $\pi_{\gamma^\pm}(x)$ and conclude that $|j'-j|$ is large. If j is much larger than j', then $\pi^{\mathcal{FF}}([x, \varphi^j o])$ is a quasigeodesic whose endpoints project onto $\pi^{\mathcal{FF}}(\{\varphi^i o\}_i)$ near $\pi^{\mathcal{FF}}\varphi^{j'} o$ and $\pi^{\mathcal{FF}}\varphi^j o$, respectively. Since $j'-j$ is large enough, this quasigeodesic crosses up long enough subsegments of $\pi^{\mathcal{FF}}(\{\varphi^i o\}_i)$ and $\pi^{\mathcal{FF}}(\gamma^+)$ that begin at $\pi^{\mathcal{FF}}(\varphi^{j'} o)$ and $\pi^{\mathcal{FF}}(x_{j'})$, respectively. Using the (B, D)-contraction at $x_{j'}^+$ of γ^+, we conclude that $[x, \varphi^j o]$ contains a point p nearby $x_{j'}^+$, which makes $d(x, \varphi^{j'} o)$ shorter than $d(x, \varphi^j o)$ and leads to a contradiction. Similar contradiction occurs due to the contracting property of γ^- at x_i^-'s when j' is much larger than j. Hence, $\pi_{\{\varphi^i o\}_i}(x)$ and $\pi_{\gamma^\pm}(x)$ are equivalent.

Now if a geodesic η on CV_N has a large projection on $\{\varphi^i o\}_i$, then it also has large projections on $\gamma^\pm$. This also forces large $\pi^{\mathcal{FF}}(\pi_{\gamma^\pm}(\eta))$, due to the argument as in $(*)$. When $\pi^{\mathcal{FF}}(\pi_{\gamma^\pm}(\eta))$ progresses in the forward direction with respect to $\{\varphi_i o\}_i$, then we employ the contracting property of γ^+ to conclude. If it progresses in the backward direction, then we employ the contracting property of γ^- to conclude. $\square$

We now explain more details on the classification of fully irreducible outer automorphisms. Coulbois and Hilion classified fully irreducibles into the following mutually distinct categories in [CH12]:

(1) *geometric* automorphisms that have geometric attracting and repelling trees,
(2) *parageometric* automorphisms that have geometric attracting tree and non-geometric attracting tree,
(3) inverses of parageometric automorphisms that have geometric repelling tree and non-geometric attracting tree, and
(4) *pseudo-Levitt* automorphisms that have non-geometric attracting and repelling trees.

Sometimes, automorphisms of category (3) and (4) are together called *ageometric* automorphisms.

In Theorem A we are concerned with fully irreducibles whose expansion factors differ from that of their inverses. Examples of such automorphisms are given in [HM07b]: parageometric fully irreducibles have expansion factors that are larger than the expansion factors of their inverses. Meanwhile, geometric fully irreducibles and their inverses have the same expansion factor, due to the analogous fact for pseudo-Anosov mapping classes. For pseudo-Levitt automorphisms, both situations can happen ([HM07b], [CH12]).

As mentioned before, fully irreducibles in $\mathrm{Out}(F_2)$ are always geometric. In contrast, [JL08] provides an example of parageometric automorphism for each $N \geq 3$. Moreover, when $N \geq 3$, if a non-elementary random walk ω on $\mathrm{Out}(F_N)$ has bounded support that generates a semigroup containing a principal fully irreducible and an inverse of a principal fully irreducible, then the probability that ω_n is pseudo-Levitt tends to 1 as $n \to \infty$ [KMPT22]. Hence, the above property of parageometric automorphisms does not imply that a generic automorphism and its inverse have different expansion factors. Despite this, we will observe in Theorem A that, for example, if the support of a non-elementary random walk ω on $\mathrm{Out}(F_N)$ generates a semigroup containing a geometric and a parageometric automorphism, then the probability that $\tau(\omega_n) = \tau(\omega_n^{-1})$ tends to 0 as $n \to \infty$.

4. Limit laws

4.1. **Limit laws and deviation inequalities.** In [Cho22b] and [Cho22c], we have established several limit laws using the pivoting technique and the existence of Schottky sets. The usage of Schottky sets were guided by the alignment lemmata for contracting axes, which are now phrased in terms of BGIP. Hence, the exactly same proofs as in [Cho22b] and [Cho22c] guarantee the following limit laws.

Proposition 4.1 ([Cho22b, Proposition 5.5])**.** *Let (X, G, o) be as in Convention 1.1 and μ be a non-elementary probability measure on G. Suppose that μ has finite p-moment for some $p > 0$. Then there exists $K > 0$ such that for any $x \in X$, we have*

$$\mathbb{E}\left[\sup_{n,n' \geq 0} (\check{\omega}_{n'} o, \omega_n\, o)_o^p\right] < K.$$

If $\check{\mu}$ also has finite p-moment, then there exists $K > 0$ such that for any $x \in X$, we have

$$\mathbb{E}\left[\sup_{n \geq 0} (x, \omega_n\, o)_o^p\right] < K, \quad \mathbb{E}\left[\sup_{n,n' \geq 0} (\check{\omega}_{n'} o, \omega_n\, o)_o^{2p}\right] < K.$$

Note that there is a slight difference between the originally stated version in [Cho22b]. This difference comes from the asymmetry of the metric. In order to establish the latter estimate with doubled exponent, we defined $\upsilon(\check{\omega}, \omega)$ in [Cho22b, Lemma 5.3] to be the minimal k such that:

(1) $\alpha := (g_{i+1}, \ldots, g_{i+M_0})$ is a Schottky sequence;
(2) $(o, \omega_i\, \Gamma(\alpha), \omega_n\, o)$ is D_1-aligned for all $n \geq k$, and

(3) $(\check{\omega}_{n'}\, o, \omega_i\, \Gamma(\alpha))$ is D_2-aligned for all $n' \geq 0$.

We then proved that the $\upsilon(\check{\omega}, \omega)$ has finite exponential moment. Similarly, we defined its dual $\check{\upsilon}(\check{\omega}, \omega)$ and observed that $(\check{\omega}_{n'} o, \omega_n\, o)_o \leq \min\{d(o, \omega_\upsilon\, o), d(o, \check{\omega}_{\check{\upsilon}} o)\}$ to prove Proposition 4.1.

Meanwhile, this time $(\check{\omega}_{n'} o, \omega_n\, o)_o$ is not bounded by $d(o, \omega_\upsilon\, o)$ but by $\frac{1}{2} d^{sym}(o, \omega_\upsilon\, o)$. This quantity can be controlled only when $\mathbb{E}_\mu[d^{sym}(o, go)^p]$ is finite, or in other words, when μ and $\check{\mu}$ have finite p-moment. Similarly, it is impossible to control $(x, \omega_n\, o)_o^p$ using $d(o, \omega_\varsigma\, o)$.

Still, it is true that $(\check{\omega}_{n'} o, \omega_n\, o)_o$ is bounded by

$$\frac{1}{2}[d(\check{\omega}_{\check{\upsilon}} o, o) + d(o, \omega_\upsilon\, o)].$$

The p-th power of this quantity can be controlled using the finite p-th moment of μ and the finite exponential moments of υ and $\check{\upsilon}$, even without a moment condition for $\check{\mu}$. Hence, we obtain the first estimate in Proposition 4.1.

Meanwhile, Gouëzel's results in [Gou21] only concerns the summation of forward progresses $\sum_i d(o, g_{n(i)} o)$ for a choice $\{n(i)\}_i$. Hence, by realizing Gouëzel's strategy using the BGIP and the concatenation lemmata in [Cho22b], we obtain the following:

Theorem B (SLLN)**.** *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. Then there exists a constant $\lambda = \lambda(\omega) \in (0, +\infty]$ such that*

$$\lim_n \frac{1}{n} d(o, \omega_n\, o) = \lambda \tag{4.1}$$

for almost every ω. Moreover, $\lambda(\mu)$ is finite if and only if μ has finite first moment.

In particular, we recover Horbez' SLLN on Outer space for non-elementary random walks ([Hor16a, Corollary 5.25]).

Theorem C (Exponential bound from below)**.** *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. Then for any $0 < L < \lambda(\mu)$, there exists $K > 0$ such that*

$$\mathbb{P}[d(o, \omega_n\, o) \leq Ln] \leq K e^{-n/K}$$

holds.

As pointed out in [Cho22b], Theorem C leads to the following corollary:

Corollary 4.2 (Large deviation principle)**.** *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. If μ has finite exponential moment, then $\{d(o, \omega_n\, o)/n\}_n$ satisfies a large deviation principle with a proper convex rate function $I : [0, +\infty) \to [0, +\infty]$ which vanishes only at $\lambda = \lambda(\mu)$.*

This is a generalization of Boulanger-Mathieu-Sert-Sisto's LDP on Gromov hyperbolic spaces.

Theorem D (CLT and its converse)**.** *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. If μ has finite second moment, then there exists a Gaussian law with variance $\sigma(\mu)^2$ to which $\frac{1}{\sqrt{n}}(d(o, \omega_n o) - n\lambda)$ converges in law. Here, λ is the escape rate of ω. Moreover, $\sigma(\mu) > 0$ if and only if μ is non-arithmetic. If μ has finite third moment or $\mu, \check{\mu}$ have finite second moment, then*

$$\limsup_{n\to\infty} \pm \frac{d(o, \omega_n o) - \lambda n}{\sqrt{2n \log\log n}} = \sigma(\mu) \quad \textit{almost surely.}$$

Conversely, suppose that μ has infinite second moment. Then for any sequence $(c_n)_n$, both $\frac{1}{\sqrt{n}}(d(o, \omega_n o) - c_n)$ and $\frac{1}{\sqrt{n}}(\tau(\omega_n) - c_n)$ do not converge in law.

The CLT for displacement on Outer space has been discussed by Horbez in [Hor18] based on Benoist-Quint's CLT for cocycles. Meanwhile, the CLT for translation length has not been discussed previously. The converses of these CLTs also seem new. The following theorem also improves the previously known results deduced from Horbez' deviation inequalities in [Hor18].

Theorem E (Geodesic tracking)**.** *Let (X, G, o) be as in Convention 1.1 and ω be the random walk generated by a non-elementary measure μ on G.*

(1) Suppose that μ, $\check{\mu}$ have finite p-th moment for some $p > 0$. Then for almost every path $\omega = (\omega_n)_n$, there exists a quasigeodesic γ such that

$$\lim_n \frac{1}{n^{1/2p}} d^{sym}(\omega_n o, \gamma) = 0.$$

(2) Suppose that μ, $\check{\mu}$ have finite exponential moment. Then there exists $K < \infty$ satisfying the following: for almost every path $\omega = (\omega_n)_n$, there exists a quasigeodesic γ such that

$$\limsup_n \frac{1}{\log n} d^{sym}(\omega_n o, \gamma) < K.$$

We now discuss the exponential genericity of fully irreducible outer automorphisms with linearly increasing translation length.

Theorem F. *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G. Let $\lambda(\omega)$ be the escape rate of ω. Then for any $0 < L < \lambda(\omega)$, there exists $K > 0$ such that*

$$\mathbb{P}\left(\omega_n \textit{ has BGIP and } \tau(\omega_n) \geq Ln\right) \geq 1 - Ke^{-n/K}$$

holds.

Recently, Kapovich, Maher, Pfaff and Taylor discussed the genericity of ageometric triangular fully irreducible outer automorphisms for random walks on $\mathrm{Out}(F_N)$ that involve a principal fully irreducible automorphism φ. A crucial step there was as follows:

Lemma 4.3 ([KMPT22, Corollary 5.14])**.** *Let γ be the lone axis of φ in CV_N and $\epsilon \geq 0$. Then there exists $R \geq 1$ such that, if ψ is a fully irreducible outer automorphism with an axis γ' such that a fraction of γ' and a translate of γ fellow travel for more than length R, then ψ is a triangular fully irreducible automorphism.*

Meanwhile, we will later observe in Subsection 4.3 that:

Proposition 4.4. *For each non-elementary probability measure μ on $\mathrm{Out}(F_N)$, there exists $K_1 > 0$ such that the following holds. Let $g \in \langle\langle \operatorname{supp}\mu \rangle\rangle$ and $M > 0$. Then there exists $K_2 > 0$ such that the following holds except for an exponentially decaying probability. For a random path $\omega = (\omega_i)_{i=1}^{\infty}$, we have $0 < i(1) < \ldots < i(K_2 n) < n$ such that $\omega_{i(1)}\, o, \omega_{i(1)}\, g^M o, \ldots, \omega_{i(K_2 n)}\, o, \omega_{i(K_2 n)}\, g^M o$ are all within K_1-neighborhood of $[o, \omega_n\, o]$ and the axis of ω_n, in order from nearest to farthest from o.*

In particular, if g was chosen to be a fully irreducible automorphism, which has BGIP, we can further argue that the axis of ω_n and a translate of the axis of g K_3-fellow travelled for long enough; here, K_3 is a constant that depends on the nature of g and μ but not on M. Note that we do not assume any moment condition here. Using this observation, we deduce the following strengthening of Kapovich-Maher-Pfaff-Taylor's theorem:

Theorem G (cf. [KMPT22, Theorem A])**.** *Let $N \geq 3$ and let μ be a non-elementary probability measure on $\mathrm{Out}(F_N)$ such that $\langle\langle \operatorname{supp}(\mu) \rangle\rangle$ contains the inverse of a principal fully irreducible automorphism. Then outside a set of exponentially decaying probability, ω_n is an ageometric triangular fully irreducible outer automorphism.*

Theorem H. *Let (X, G, o) be as in Convention 1.1, and ω be the random walk generated by a non-elementary measure μ on G.*

(1) If μ has finite p-th moment for some $p > 0$, then

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} [d(o, \omega_n\, o) - \tau(\omega_n)] = 0 \quad a.s.$$

(2) If μ, $\check{\mu}$ have finite p-th moment for some $p > 0$, then

$$\lim_{n\to\infty} \frac{1}{n^{1/2p}} [d(o, \omega_n\, o) - \tau(\omega_n)] = 0 \quad a.s.$$

(3) If μ, $\check{\mu}$ have finite first moment, then there exists $K > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{\log n} [d(o, \omega_n\, o) - \tau(\omega_n)] \leq K \quad a.s.$$

Again, the estimate in (1) without doubled exponent is due to the lack of moment conditions for $\check{\mu}$; we bound the discrepancy with $\frac{1}{2}[d(\check{\omega}_{\check{v}} o, o) + d(o, \omega_v\, o)]$, not $\min\{d(o, \omega_v\, o), d(o, \check{\omega}_{\check{v}} o)\}$.

Theorem I. *Let (X, G, o) be as in Convention 1.1 and suppose that G is finitely generated. Then for each $k > 0$, there exists a finite generating set S of G such that*

$$\frac{\#\left\{(g_1, \ldots, g_k) : \begin{array}{c} g_1, \ldots, g_k \in B_n(e), \langle g_1, \ldots, g_k \rangle \text{ is q.i. embedded} \\ \text{into a quasi-convex subset of } X \end{array}\right\}}{(\# B_n(e))^k}$$

converges to 1 exponentially fast.

Theorem J. *Let (X, G, o) be as in Convention 1.1, and $\omega^{(1)}, \ldots, \omega^{(k)}$ be k independent random walk generated by a non-elementary measure μ on G. Then there exists $K > 0$ such that*

$$\mathbb{P}\left[\langle \omega_n^{(1)}, \ldots, \omega_n^{(k)} \rangle \text{ is q.i. embedded into a quasi-convex subset of } X\right] \geq 1 - K e^{-n/K}.$$

Having recalled these results, we now prepare the machinery of pivoting technique to prove our main theorem.

4.2. **Pivotal times and pivoting.** In this subsection, we recall the pivoting technique developed in [Gou21], [BCK21] and [Cho22b]. For a complete proof, refer to Section 4 of [Cho22b].

Let $(w_i)_{i=0}^{\infty}$, $(v_i)_{i=1}^{\infty}$ be isometries in G. Now given a sequence

$$s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_n, \beta_n, \gamma_n, \delta_n) \in S^{4n},$$

we first define

$$a_i := \Pi(\alpha_i),\ b_i := \Pi(\beta_i)\ c_i := \Pi(\gamma_i),\ d_i := \Pi(\delta_i). \tag{4.2}$$

We then consider isometries that are subwords of

$$w_0 a_1 b_1 v_1 c_1 d_1 w_1 \cdots a_k b_k v_k c_k d_k w_k \cdots .$$

More precisely, we set the initial case $w_{-1,2}^{+} := id$ and define

$$\begin{aligned} w_{i,2}^{-} &:= w_{i-1,2}^{+} w_{i-1}, & w_{i,1}^{-} &:= w_{i,2}^{-} a_i, & w_{i,0}^{-} &:= w_{i,2}^{-} a_i b_i, \\ w_{i,0}^{+} &:= w_{i,2}^{-} a_i b_i v_i, & w_{i,1}^{+} &:= w_{i,2}^{-} a_i b_i v_i c_i, & w_{i,2}^{+} &:= w_{i,2}^{-} a_i b_i v_i c_i d_i. \end{aligned}$$

We also employ notations

$$\begin{aligned} \Upsilon(\alpha_i) &:= w_{i,2}^{-} \Gamma(\alpha_i), & \Upsilon(\beta_i) &:= w_{i,1}^{-} \Gamma(\beta_i), \\ \Upsilon(\gamma_i) &:= w_{i,0}^{+} \Gamma(\gamma_i), & \Upsilon(\delta_i) &:= w_{i,1}^{+} \Gamma(\delta_i). \end{aligned}$$

for simplicity.

We then defined the set $P_n = P_n(s, (w_i)_i, (v_i)_i) \subseteq \{1, \ldots, n\}$. Our main estimates were as follows.

Lemma 4.5 ([Cho22b, Lemma 4.1]). *Let $P_n = \{i(1) < \ldots < i(m)\}$. Then*

$$\left(o, \Upsilon(\alpha_{i(1)}), \Upsilon(\beta_{i(1)}), \Upsilon(\gamma_{i(1)}), \Upsilon(\delta_{i(1)}), \ldots, \Upsilon(\alpha_{i(m)}), \Upsilon(\beta_{i(m)}), \Upsilon(\gamma_{i(m)}), \Upsilon(\delta_{i(m)}), y_{n+1,2}^{-}\right)$$

is a subsequence of a D_0-aligned sequence of Schottky axes. In particular, it is D_1-aligned.

In [Cho22b], we have observed a sufficient condition for $P_k = P_{k-1} \cup \{k\}$ to hold. Namely, the conditions

$$\text{diam}\left(\pi_{\Upsilon(\gamma_k)}(y_{k,0}^-) \cup y_{k,0}^+\right) = \text{diam}\left(\pi_{\Gamma(\gamma_k)}(v_k^{-1}o) \cup o\right) < K_0, \tag{4.3}$$

$$\text{diam}\left(\pi_{\Upsilon(\delta_k)}(y_{k+1,2}^-) \cup y_{k,2}^+\right) = \text{diam}\left(\pi_{\Gamma^{-1}(\delta_k)}(w_k o) \cup o\right) < K_0, \tag{4.4}$$

$$\text{diam}\left(\pi_{\Upsilon(\beta_k)}(y_{k,1}^+) \cup y_{k,0}^-\right) = \text{diam}\left(\pi_{\Gamma^{-1}(\beta_k)}(v_k c_k o) \cup o\right) < K_0, \tag{4.5}$$

$$\text{diam}\left(\pi_{\Upsilon(\alpha_k)}(z_{k-1}) \cup y_{k,2}^-\right) = \text{diam}\left(\pi_{\Gamma(\alpha_k)}\left((w_{k,2}^-)^{-1} z_{k-1}\right) \cup o\right) < K_0. \tag{4.6}$$

guaranteed the addition of k to the set of pivotal times. Each condition excluded at most one element out of the Schottky set S and we obtained:

Lemma 4.6 ([Cho22b, Lemma 4.2])**.** *For $1 \le k \le n$, $s \in S^{4(k-1)}$, we have*

$$\mathbb{P}\left(\#P_k(s, \alpha_k, \beta_k, \gamma_k, \delta_k) = \#P_{k-1}(s) + 1\right) \ge 1 - 4/N_0.$$

Given $\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}, \delta_{k-1}$, we define the set $\tilde{S}_k$ of triples $(\alpha_k, \beta_k, \gamma_k)$ in S^3 that satisfy Condition 4.3, 4.5 and 4.6. We then observed that $\#\left[S^3 \setminus \tilde{S}_k\right] \le 3(\#S)^2$. Moreover, for $(\alpha_k, \beta_k, \gamma_k) \in \tilde{S}_k$, $\{(\alpha_k, \beta_k', \gamma_k) \in \tilde{S}_k : \beta_k \in S\}$ has at least $\#S - 1$ elements. We finally had:

Lemma 4.7 ([Cho22b, Lemma 4.3])**.** *Let $i \in P_k(s)$ for a choice $s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_n, \beta_n, \gamma_n, \delta_n)$, and $\bar{s}$ be obtained from s by replacing $(\alpha_i, \beta_i, \gamma_i)$ with*

$$(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i) \in \tilde{S}_i(\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}, \delta_{i-1}).$$

Then $P_l(s) = P_l(\bar{s})$ and $\tilde{S}_l(s) = \tilde{S}_l(\bar{s})$ for each $1 \le l \le k$.

Given $1 \le k \le n$ and a partial choice $s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_k, \beta_k, \gamma_k, \delta_k)$, we defined pivoting as follows: $\bar{s} = (\bar{\alpha}_1, \bar{\beta}_1, \bar{\gamma}_1, \bar{\delta}_1, \ldots, \bar{\alpha}_k, \bar{\beta}_k, \bar{\gamma}_k, \bar{\delta}_k)$ is *pivoted from s* if:

- $\delta_j = \bar{\delta}_j$ for all $1 \le j \le k$,
- $(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i) \in \tilde{S}_i(s)$ for each $i \in P_k(s)$, and
- $(\alpha_j, \beta_j, \gamma_j) = (\bar{\alpha}_j, \bar{\beta}_j, \bar{\gamma}_j)$ for each $j \in \{1, \ldots, k\} \setminus P_k(s)$.

Lemma 4.7 then asserted that being pivoted from each other is an equivalence relation.

Corollary 4.8. *When $s = (\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^n$ is chosen from S^{4n} with the uniform measure, $\#P_n(s)$ is greater in distribution than the sum of n i.i.d. X_i, whose distribution is given by*

$$\mathbb{P}(X_i = j) = \begin{cases} (N_0 - 4)/N_0 & \text{if } j = 1, \\ (N_0 - 4)4^{-j}/N_0^{-j+1} & \text{if } j < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

More generally, the distribution of $\#P_{k+n}(s) - \#P_k(s)$ *conditioned on the choices of* $(\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^k$ *also dominates the sum of* n *i.i.d.* X_i.

Moreover, we have $\mathbb{P}(\#P_n(s) \le (1 - 10/N_0)n) \le e^{-Kn}$ *for some* $K > 0$.

We now recall the pivoting for translation length in [Cho22c, Subsection 4.2]. We fix an equivalence class $\mathcal{E} \subseteq S^{4n}$ made by pivoting. $\mathcal{E}$ has a well-defined set of pivotal times $P_n(\mathcal{E}) = \{i(1), \ldots, i(M)\}$, and a choice $s \in \mathcal{E}$ is determined by the choices $(\alpha_{i(l)}, \beta_{i(l)}, \gamma_{i(l)})_{l=1}^M$. We also denote $w_{n+1,2}^-(s)$ by w for convenience.

Let

$$\begin{aligned}\phi_k &:= (w_{i(M-k+1),0}^-)^{-1} w w_{i(k),2}^- \\ &= v_{i(M-k+1)} c_{i(M-k+1)} d_{i(M-k+1)} w_{i(M-k+1)} \cdots a_n b_n v_n c_n d_n w_n \\ &\quad \cdot w_0 a_1 b_1 v_1 c_1 d_1 w_1 \cdots a_{i(k)-1} b_{i(k)-1} v_{i(k)-1} c_{i(k)-1} d_{i(k)-1} w_{i(k)-1}\end{aligned}$$

for $1 \le k \le \lfloor M/2 \rfloor$ and

$$S_k^*(s) := \left\{ \alpha_{i(k)} \in S \quad : \left(w^{-1} y_{i(M-k+1),0}^-,\ \Upsilon(\alpha_{i(k)}) \right) \text{ is } K_0\text{-aligned} \right\},$$

$$S_{M-k+1}^*(s) := \left\{ \beta_{i(M-k+1)} \in S : \left(w^{-1}\, \Upsilon(\beta_{i(M-k+1)}), y_{i(k),1}^- \right) \text{ is } K_0\text{-aligned} \right\}.$$

We observed that:

Lemma 4.9. *Let* $1 \le k \le M/2$. *Suppose that* $s = (\alpha_{i(l)}, \beta_{i(l)}, \gamma_{i(l)})_{l=1}^M \in \mathcal{E}_n$ *satisfies*

$$\alpha_{i(k)} \in S_k^*(s), \quad \beta_{i(M-k+1)} \in S_{M-k+1}^*(s).$$

Then $w = w_{n+1,2}^-$ *is a contracting isometry. Moreover,*

$$\left(\ldots, w^{-1}\, \Upsilon(\beta_{i(M-k+1)}), \Upsilon(\alpha_{i(k)}), \Upsilon(\beta_{i(k)}), \Upsilon(\gamma_{i(k)}), \ldots, \Upsilon(\beta_{i(M-k+1)}), w\, \Upsilon(\alpha_{i(k)}), \ldots\right)$$

is a subsequence of a D_1*-aligned sequence of Schottky axes.*

Given a choice

$\bar{s} = (\bar{\alpha}_{i(l)}, \bar{\beta}_{i(l)}, \bar{\gamma}_{i(l)})_{l=1,\ldots,k-1,M-k+2\ldots,M} \in \tilde{S}_{i(1)} \times \cdots \times \tilde{S}_{i(k-1)} \times \tilde{S}_{i(M-k+2)} \times \cdots \times \tilde{S}_{i(M)}$,

we define

$$S_k^\dagger := \left\{ \begin{array}{c} (\alpha_{i(k)}, \beta_{i(k)}, \gamma_{i(k)}, \alpha_{i(M-k+1)}, \beta_{i(M-k+1)}, \gamma_{i(M-k+1)}) \in \tilde{S}_{i(k)} \times \tilde{S}_{i(M-k+1)} \\ : \quad \alpha_{i(k)} \in S_k^*(\bar{s}, \gamma_{M-k+1}) \text{ and} \\ \beta_{i(M-k+1)} \in S_{M-k+1}^*(\bar{s}, \alpha_{i(k)}, \gamma_{i(M-k+1)}) \end{array} \right\}$$

Then we proved the following:

Lemma 4.10. *For each* $1 \le k \le \lfloor M/2 \rfloor$, *the cardinality of* $\tilde{S}_k^\dagger$ *is at least* $(\#S)^6 - 8(\#S)^5$.

Let us finally mention a variant of the usual pivoting called *v-pivoting.* We fix subsets $S_1, S_2 \subseteq S$ of cardinality at least $N_0/4$, and a subset $A \subseteq G$. We then assume that for each $s_1 \in S_1, s_2 \in S_2$ and $v \in A$, the two sequences

$$\left(v^{-1}o, \Gamma(s_2)\right), \quad \left(v\Pi(s_2)o, \Gamma^{-1}(s_1)\right) \tag{4.8}$$

are K_0-aligned.

As in Subsection 4.2, we consider the subwords of

$$w_0 a_1 b_1 v_1 c_1 d_1 \cdots a_n b_n v_n c_n d_n w_n \cdots$$

and define $w_{i,j}^{\pm}$, $y_{i,j}^{\pm}$ analogously. This time, however, w_i's are chosen from G and *v_i's are chosen from A*. Also, we will not fix the choice of $(v_i)_i$ this time; only $(w_i)_i$ is fixed. Also, α_i, β_i's are chosen from S_1 and γ_i, δ_i's are chosen from S_2. In other words, a choice $s = (\alpha_1, \beta_1, \ldots, \gamma_n, \delta_n)$ is drawn from $(S_1^2 \times S_2^2)^n$. We could still define the set of pivotal times based on the same criteria with Subsection 4.2, and observed:

Lemma 4.11. *Let $i \in P_k(s, \mathbf{v})$ for a choice $s = (\alpha_1, \ldots, \delta_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$. If $\mathbf{v}' = (v_1', \ldots, v_n')$ is made from $\mathbf{v}$ by replacing v_i with an element of A, then $P_l(s, \mathbf{v}) = P_l(s, \mathbf{v}')$ and $\tilde{S}_l(s, \mathbf{v}) = \tilde{S}_l(s, \mathbf{v}')$ for each $1 \le l \le k$.*

Given a choice $s = (\alpha_1, \ldots, \delta_n) \in (S_1^2 \times S_2^2)^n$ and $\mathbf{v} = (v_i)_{i=1}^n \in A^n$, we declared that $(s, \mathbf{v}')$ is *v-pivoted from* $(s, \mathbf{v})$ if $\mathbf{v}'$ differs from $\mathbf{v}$ only at the pivotal times for $(s, \mathbf{v})$. Then Lemma 4.11 tells us that being v-pivoted from each other is an equivalence relation that preserves the set of pivotal times.

4.3. **Proof of Theorem A.** We are now ready to prove Theorem A.

Proof. Let us first fix a positive integer n'. Since μ is asymptotically asymmetric, we can take $\alpha = (\phi_1, \ldots, \phi_{M'}), \beta = (\varphi_1, \ldots, \varphi_{M'}) \in (\operatorname{supp} \mu)^{M'}$ with $\Pi(\alpha) = \phi$, $\Pi(\beta) = \varphi$ such that

$$L := \left[\tau(\phi) - \tau(\phi^{-1})\right] - \left[\tau(\varphi) - \tau(\varphi^{-1})\right] > 0.$$

Let also

$$L_1^{\pm} := d(o, \phi^{\pm 1} o), \quad L_2^{\pm} := d(o, \varphi^{\pm 1} o)$$

By taking self-concatenations of α and β if necessary, we may assume that $(L_1^+ - L_1^-) - (L_2^+ - L_2^-) \ge L/2 \ge 8E_0 n'$.

At the moment, there exist at least $N_0 - 2$ Schottky choices $s_2 \in S$ such that $(\phi^{-1} o, \Gamma(s_2))$ and $(\varphi^{-1} o, \Gamma(s_2))$ are K_0-aligned. Choose $N_0/3$ of them and label as $s_2^{(1)}, \ldots, s_2^{(N_0/3)}$. We now seek choices $s_1 \in S$ such that $(\phi\, \Pi(s_2^{(i)}) o, \Gamma^{-1}(s_1))$ and $(\varphi\, \Pi(s_2^{(i)}) o, \Gamma^{-1}(s_1))$ are K_0-aligned for each $i = 1, \ldots, N_0/3$. Since these are $2N_0/3$ conditions in total, there exist at least $N_0/3$ Schottky choices realizing them. In summary, there exist subsets $S_1, S_2 \subseteq S$ of cardinality $N_0/3$ such that

$$\left(\phi^{-1} o, \Gamma(s_2)\right), \left(\varphi^{-1} o, \Gamma(s_2)\right), \left(\Gamma(s_1), \Pi(s_1) \phi\, \Pi(s_2^{(i)}) o\right), \left(\Gamma(s_1), \Pi(s_1) \varphi\, \Pi(s_2^{(i)}) o\right)$$

are K_0-aligned for all $s_1 \in S_1$ and $s_2 \in S_2$.

Let μ' be the measure assigning $1/2$ to each of α and β. We then consider the decomposition

$$\mu^{4M_0 + M'} = \alpha \left(\mu_{S_1}^2 \times \mu' \times \mu_{S_2}^2\right) + (1 - \alpha)\nu$$

for some $0 < \alpha < 1$ and ν. Here, we perform the same procedure as in the proof of Proposition 4.8 in [Cho22c]. We then consider:

- Bernoulli RVs ρ_i with $\mathbb{P}(\rho_i = 1) = \alpha$ and $\mathbb{P}(\rho_i = 0) = 1 - \alpha$,
- η_i with the law $\mu_{S_1}^2 \times \mu' \times \mu_{S_2}^2$, and
- ν_i with the law ν,

all independent, and define

$$(g_{(4M_0+1)k+1}, \ldots, g_{(4M_0+1)(k+1)}) = \begin{cases} \nu_k & \text{when } \rho_k = 0, \\ \eta_k & \text{when } \rho_k = 1. \end{cases}$$

Then $(g_i)_{i=1}^{\infty}$ has the law μ^{∞}. We now define Ω to be the ambient probability space on which the above RVs are all measurable. We will denote an element of Ω by ω. We also fix

- $\omega_k := g_1 \cdots g_k$,
- $\mathscr{B}(k) := \sum_{i=0}^{k} \rho_i$, i.e., the number of the Schottky slots till k, and
- $\vartheta(i) := \min\{j \geq 0 : \mathscr{B}(j) = i\}$, i.e., the i-th Schottky slot.

For each $\omega \in \Omega$ and $i \geq 1$ we define

$$\begin{aligned} w_{i-1} &:= g_{4M_0[\vartheta(i-1)+1]+1} \cdots g_{4M_0\,\vartheta(i)}, \\ \alpha_i &:= (g_{4M_0\,\vartheta(i)+1}, \ldots, g_{4M_0\,\vartheta(i)+M_0}), \\ \beta_i &:= (g_{4M_0\,\vartheta(i)+M_0+1}, \ldots, g_{4M_0\,\vartheta(i)+2M_0}), \\ v_i &:= g_{4M_0\,\vartheta(i)+2M_0+1}, \\ \gamma_i &:= (g_{4M_0\,\vartheta(i)+2M_0+2}, \ldots, g_{4M_0\,\vartheta(i)+3M_0+1}), \\ \delta_i &:= (g_{4M_0\,\vartheta(i)+3M_0+2}, \ldots, g_{4M_0\,\vartheta(i)+4M_0+1}). \end{aligned}$$

In other words, $\eta_{\vartheta(i)}$ corresponds to $(\alpha_i, \beta_i, v_i, \gamma_i, \delta_i)$ and w_i corresponds to the products of intermediate steps ν_k's in between $\eta_{\vartheta(i-1)}$ and $\eta_{\vartheta(i)}$. As in Section 4.2, we employ the notation $a_i := \Pi(\alpha_i)$, $b_i := \Pi(\delta_i)$ and so on.

In order to represent ω_n for arbitrary n, we set $n' := \lfloor n/4M_0 \rfloor - 1$ and $w^{(n)} := g_{4M_0[\vartheta(\mathscr{B}(n'))+1]+1} \cdots g_n$. We then have

$$\omega_n = w_0 a_1 b_1 v_1 c_1 d_1 w_1 \cdots a_{\mathscr{B}(n')} b_{\mathscr{B}(n')} c_{\mathscr{B}(n')} d_{\mathscr{B}(n')} w^{(n)} \tag{4.9}$$

Here, we first fix the choices of ρ_i's and ν_i's; this determines $\mathscr{B}(n')$ and the isometries $(w_0, \ldots, w^{(n)}), (v_1, \ldots, v_n)$. Then we consider the set of pivotal times $P_{\mathscr{B}(n')}(s)$ for $s \in (S_1^{(2)} \times S_2^{(2)})^n$. After this process, we define

$$\mathcal{P}_n(\omega) := \left\{ (4M_0 + 1)\,\vartheta(i) : i \in P_{\mathscr{B}(n')}(s) \right\}.$$

Note that $\mathscr{B}(n')$ is a sum of i.i.d.s of Bernoulli distribution: it is linearly increasing outside a set of exponential probability. Moreover, $\#P_{\mathscr{B}(n')}$ is linearly increasing with respect to $\mathscr{B}(n')$ in the sense of Corollary 4.8. Hence, there exists $K > 0$ such that $\mathcal{P}_n(\omega) \geq 10Kn$ outside a set of exponentially decaying probability.

At the moment, we consider an equivalence class $\mathcal{E}$ of n-step paths with $M \geq 2Kn + n'$ pivotal times. Here, the equivalence relation is made by the usual pivoting at the first and the last Kn pivotal times, and by the v-pivoting at the $(Kn + 1)$-th, $\ldots$, $(Kn + n')$-th pivotal times. By the previous observation, such equivalence classes take up all cases except a set of exponentially decaying probability. Moreover, most pivotal choices at the first and the last Kn pivotal times will lead to $\alpha_{i(k)} \in S_k^*(\omega), \beta_{i(M-k+1)} \in S_{M-k+1}^*(\omega)$ for some $1 \le k \le Kn$. We stick to such choices and give up a set of probability $(8/N_0)^{Kn}$.

Now we are discussing the situation on a finer equivalence class $\mathcal{E}_1 \subseteq \mathcal{E}$ made by v-pivoting at the $(Kn+1)$-th, ..., $(Kn+n')$-th pivotal times. Each choice in $\mathcal{E}_1$ can be recorded by $\epsilon = (\epsilon_l)_{l=1}^{n'} \in \{0,1\}^{n'}$, where $v_{i(l)} = \phi$ if $\epsilon_l = 0$ and $v_{i(l)} = \varphi$ if $\epsilon_l = 1$. Let us now define

$$D_+ := \left[d\left(\omega_n^{-1} y^+_{i(Kn+n'),0}, y^-_{i(Kn+1),0}\right) + \sum_{l=1}^{n'-1} d\left(y^+_{i(Kn+l),0}, y^-_{i(Kn+l+1,0),0}\right)\right],$$
$$D_- := \left[d\left(y^-_{i(Kn+1),0}, \omega_n^{-1} y^+_{i(Kn+n'),0}\right) + \sum_{l=1}^{n'-1} d\left(y^-_{i(Kn+l+1,0),0}, y^+_{i(Kn+l),0}\right)\right].$$

Note that D_+, D_- do not depend on the choices $\{v_{i(l)}\}_{l=Kn+1}^{Kn+n'}$; these are invariant across $\mathcal{E}_1$. Moreover, Lemma 4.9 and Corollary 2.20 guarantee that:

$$\left|\tau(\omega_n) - D_+ - L_1^+\left(n' - \sum_{l=1}^{n'} \epsilon_l\right) - L_2^+ \sum_{l=1}^{n'} \epsilon_l\right| = \left|\tau(\omega_n) - D_+ - \sum_{l=1}^{n'} d(o, v_{i(l)} o)\right| \le 2n'E_0.$$

Similarly, we have

$$\left|\tau(\omega_n^{-1}) - D_- - L_1^-\left(n' - \sum_{l=1}^{n'} \epsilon_l\right) - L_2^- \sum_{l=1}^{n'} \epsilon_l\right| = \left|\tau(\omega_n^{-1}) - D_- - \sum_{l=1}^{n'} d(o, v_{i(l)} o)\right| \le 2n'E_0.$$

These two implies that

(4.10)
$$\left|\left[\tau(\omega_n) - \tau(\omega_n^{-1})\right] - (D_+ - D_-) - (L_1^+ - L_1^-)n' + L \sum \epsilon\right| \le 4n'E_0.$$

At the moment, suppose that $\omega \in \mathcal{E}_1$ satisfies $|\tau(\omega_n) - \tau(\omega_n^{-1})| < n'E_0$. Then Equation 4.10 tells us that $|\tau(\tilde{\omega}_n) - \tau(\tilde{\omega}_n^{-1})| > n'E_0$ for any other $\tilde{\omega} \in \mathcal{E}_1$ such that $\sum \epsilon(\omega) \neq \sum \epsilon(\tilde{\omega})$. Note that on $\mathcal{E}_1$, $\sum \epsilon$ follows the binomial distribution: $\mathbb{P}(\sum \epsilon = k) = 2^{-n'}\binom{n'}{k}$. Since $\max_k 2^{-n'}\binom{n'}{k}$ is $O(1/\sqrt{n'})$, the conditional probability is controlled as

$$\mathbb{P}\left(|\tau(\omega_n) - \tau(\omega_n^{-1})| < n'E_0 \,\middle|\, \mathcal{E}_1\right) < C/\sqrt{n'} \tag{4.11}$$

for some $C > 0$. We then sum up the conditional probabilities for these $\mathcal{E}_1$ and the remaining exponentially decaying probability to deduce

$$\mathbb{P}\left(|\tau(\omega_n) - \tau(\omega_n^{-1})| < n'E_0\right) < 2C/\sqrt{n'}$$

for large enough n. We now send n' to infinity to conclude. □

The previous proof also implies Proposition 4.4. One can pick some suitable M' and $\alpha = \beta \in (\operatorname{supp} \mu)^{M'}$ such that $g^M = \phi = \Pi(\alpha) = \varphi = \Phi(\beta)$.

Then the linear growth of the set of pivotal times, Lemma 4.5 and Corollary 2.20 imply the desired conclusion.

5. Further questions

In contrast to the exponential genericity in Theorem G, Theorem A only asserts that the unwanted probability converges to 0 and does not discuss the decay rate. This is due to the nature of our argument in the proof: Inequality 4.11 followed from the estimation of a single binomial coefficient, which does not decay exponentially.

In principle, the proof of Proposition G relied on the appearance of a single geodesic segment along the axis of a random isometry. In other words, the triangularity of a random isometry was due to the stability of the lone axis of a principal fully irreducible automorphism. In the proof of A, we observed the opposite situation; regardless of what happened in the beginning or the ending part of $[o, \omega_n o]$, the v-pivotal choices made at the intermediate pivotal times can cause the asymmetry of the forward and backward expansion factors. In other words, the stability of certain geodesic segments cannot rescue the situation. Rather, the asymmetry arises from the cumulative effect of two types of progresses that differ in the differences between forward and backward distances.

Considering this, a refined argument may lead to the CLT for the difference between forward and backward expansion factors. For the moment, we only content ourselves with the following question:

Question 5.1. *Is the mismatch of forward and backward expansion factors of an outer automorphism exponentially generic?*

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