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타이히뮐러 공간 위에서의 무작위 행보에 관한 이론

Theory of random walks on Teichmüller space

2023

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한 국 과 학 기 술 원

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Theory of random walks on Teichmüller space

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The study was conducted in accordance with Code of Research Ethics[1].

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한글 초록: 본 논문에서는 유한 종류인 쌍곡 곡면의 타이히뮐러 공간 위에서의 무작위 행보를 다룬다. 특히, 무작위 행보에 관한 극한 법칙 중 큰 수의 법칙, 중심극한정리, 측지선 따라감 등을 가장 일반적인 모멘트 조건 하에서 확립한다. 이 결과들의 응용으로서 일반적인 사상류의 성질을 탐구하는데, 그 일례로 특정 생성 집합을 기준으로 사상류군에서 유사-아나사브 사상류가 지수함수적으로 일반적임을 보인다. 또, 타이히뮐러 공간과 비슷한 성질을 공유하는 다른 공간에서의 무작위 행보에 대해서도 같은 이론을 적용해 극한 법칙들을 도출한다.

핵심 낱말: 무작위 행보, 타이히뮐러 공간, 유사-아나사브 사상류, 큰 수의 법칙, 중심극한정리, 측지선 따라가기

Abstract: We study random walks on the Teichmüller space of a hyperbolic surface of finite type. In particular, we establish limit laws on random walks including the laws of large numbers, central limit theorem and geodesic tracking under the optimal moment conditions. As an application, we investigate the property of a generic mapping class. In particular, we show that pseudo-Anosov mappings are exponentially generic in the mapping class group with respect to certain generating sets. Finally, we deduce analogous limit laws for random walks on other spaces that share a similar geometric property with Teichmüller space.

Keywords: Random walk, Teichmüller space, Pseudo-Anosov mapping class, Law of large numbers, Central limit theorem, Geodesic tracking

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Chapter 1. Introduction

Throughout, we fix a hyperbolic surface of finite type Σ. The space of our interest is the Teichmüller space $\mathcal{T}(\Sigma)$ of Σ, which consists of equivalence classes of marked hyperbolic structures (or conformal structures) on Σ. Here, two structures are equivalent if they are homotopic relative to the marking. $\mathcal{T}(\Sigma)$ is equipped with two canonical metrics, namely, the Teichmüller metric $d_{\mathcal{T}}$ and the Weil-Petersson metric d_{WP}.

The group associated with $\mathcal{T}(\Sigma)$ is the mapping class group $\mathrm{Mod}(\Sigma)$ of Σ, the collection of orientation-preserving self-homeomorphisms of Σ up to homotopy. Mapping classes of Σ naturally act on $\mathcal{T}(\Sigma)$ as isometries with respect to both $d_{\mathcal{T}}$ and d_{WP}. Moreover, the action of a mapping class φ on $\mathcal{T}(\Sigma)$ reveals its dynamical property on Σ. In particular, the celebrated Nielsen-Thurston's theorem classifies mapping classes into three categories, i.e., periodic, reducible and pseudo-Anosov mapping classes, based on the dynamics of their actions on the Thurston compactifiaction of $\mathcal{T}(\Sigma)$. Among three categories, pseudo-Anosov mapping classes exhibit the most complicated dynamics. Thanks to many pioneering results, Teichmüller space and the mapping class group have become central objects in low-dimensional topology and geometry, playing a key role in 3-manifold theory, hyperbolic geometry and geometric group theory.

Combining the author's contributions in [BCK21], [Cho21a], [Cho21b] and [Cho22], this dissertation aims to present a systematic study of random walks on Teichmüller space and its analogues. More precisely, we consider i.i.d.s $g_1, g_2, \ldots$ on $\mathrm{Mod}(\Sigma)$ and investigate the asymptotic behavior of the n-th step mapping class $\omega_n := g_1 g_2 \cdots g_n$. Each mapping class $g \in \mathrm{Mod}(\Sigma)$ is associated with two dynamical quantities, the displacement $d(o, go)$ of o by g and the translation length $\tau(g) := \lim_n \frac{1}{n} d(o, g^n o)$ of g. We establish various limit laws for these quantities, including strong laws of large numbers (SLLNs), central limit theorems (CLTs) and laws of the iterated logarithm (LILs).

This systematic study leads to a fruitful geometric understanding of the mapping class group and its action on Teichmüller space. In particular, we establish the geodesic tracking of random walks and the genericity of pseudo-Anosov mapping classes. Let us emphasize the importance of the latter result in particular. We have several recipes for pseudo-Anosov mapping classes and the presence of pseudo-Anosovs led to a deeper understanding of the group structure of $\mathrm{Mod}(\Sigma)$, including Tits alternative and the rigidity theorems. It was then further conjectured that pseudo-Anosov mapping classes are generic in the mapping class group. There are two possible ways to formulate this genericity: one is to observe the asymptotics of random walks, and the other is to pick an element from the ball of radius n in the Cayley graph.

Together with Hyungryul Baik and Dongryul M. Kim, we proved in [BCK21] that non-elementary random walks on the mapping class group eventually become pseudo-Anosov almost surely. This result generalizes the previuos ones by Joseph Maher. Namely, Maher proved that random walks become pseudo-Anosov in probability in [Mah11]. Moreover, random walks with bounded support on the curve complex become pseudo-Anosov almost surely [Mah12]. The methodology explained in this dissertation also has some new flavor. We deduce the genericity of pseudo-Anosov mapping classes via Teichmüller geometry, whereas all previously known arguments rely on either the geometry of the curve complex or the homology representation of the mapping class group [Riv08].

Further, in [Cho21b], we also proved that pseudo-Anosovs predominate large balls in the Cayley graph for particular choices of generating set. This answers a version of a long-standing conjecture by

Benson Farb [Far06]. The methodology involved here also applies to the outer automorphism group and CAT(0) groups; this is outlined in [Cho22].

Our main results follow.

Theorem A (SLLN)**.** *Let ω be the random walk on* $\mathrm{Mod}(\Sigma)$ *generated by a non-elementary measure μ. Then there exists a constant $\lambda = \lambda(\mu) \in (0, +\infty]$ such that*

$$\lim_n \frac{1}{n} d(o, \omega_n o) = \lim_n \frac{1}{n} \tau(\omega_n) = \lambda \tag{1.0.1}$$

for almost every ω. Moreover, $\lambda(\mu)$ is finite if and only if μ has finite first moment.

We call $\lambda(\mu)$ in Theorem A the *escape rate* of μ.

Theorem B. *Let ω be the random walk on* $\mathrm{Mod}(\Sigma)$ *generated by a non-elementary measure μ. If μ has finite first moment, then there exists $K > 0$ such that*

$$\limsup_{n\to\infty} \frac{1}{\log n} |d(o, \omega_n o) - \tau(\omega_n)| \le K \quad a.s.$$

Theorem C (CLT and LIL)**.** *Let ω be the random walk on* $\mathrm{Mod}(\Sigma)$ *generated by a non-elementary measure μ. If μ has finite second moment, then there exists a Gaussian law with variance $\sigma(\mu)^2$ to which $\frac{1}{\sqrt{n}}(d(o, \omega_n o) - n\lambda)$ and $\frac{1}{\sqrt{n}}(\tau(\omega_n) - n\lambda)$ converge in law. Here, $\sigma(\mu) > 0$ if and only if μ is non-arithmetic. Moreover, we have*

$$\limsup_{n\to\infty} \pm \frac{d(o, \omega_n o) - \lambda n}{\sqrt{2n \log\log n}} = \limsup_{n\to\infty} \pm \frac{\tau(\omega_n) - \lambda n}{\sqrt{2n \log\log n}} = \sigma(\mu) \quad \textit{almost surely.}$$

Conversely, suppose that μ has infinite second moment. Then for any sequence $(c_n)_n$, both $\frac{1}{\sqrt{n}}(d(o, \omega_n o) - c_n)$ and $\frac{1}{\sqrt{n}}(\tau(\omega_n) - c_n)$ do not converge in law.

Theorem D (Genericity of pseudo-Anosovs I)**.** *Let ω be the random walk on* $\mathrm{Mod}(\Sigma)$ *generated by a non-elementary measure μ. Let $\lambda = \lambda(\mu)$ be the escape rate of μ and $0 < L < \lambda$. Then there exists $K > 0$ such that*

$$\mathbb{P}\big(\omega_n \textit{ is a pseudo-Anosov with } \tau(\omega_n) \ge Ln\big) \ge 1 - Ke^{-n/K}$$

holds for all n.

Theorem E (Geodesic tracking)**.** *Let ω be the random walk on* $\mathrm{Mod}(\Sigma)$ *generated by a non-elementary measure μ.*

1. *Suppose that μ has finite p-th moment for some $p > 0$. Then for almost every path $\omega = (\omega_n)_n$, there exists a quasigeodesic γ such that*

$$\lim_n \frac{1}{n^{1/2p}} d(\omega_n o, \gamma) = 0.$$

2. *Suppose that μ has finite exponential moment. Then there exists $K < \infty$ satisfying the following: for almost every path $\omega = (\omega_n)_n$, there exists a quasigeodesic γ such that*

$$\limsup_n \frac{1}{\log n} d(\omega_n o, \gamma) < K.$$

Theorem F (Genericity of pseudo-Anosovs II)**.** *Let G be a finitely generated non-elementary subgroup of* $\mathrm{Mod}(\Sigma)$*. Then there exists a finite generating set $S \subseteq G$ such that the proportion of non-pseudo-Anosov mapping classes in the ball $B_S(n)$ decays exponentially as $n \to \infty$.*

These results have been partially observed by other authors. The point of this dissertation is to weaken the assumptions by employing a method that applies to a wide range of spaces. We now explain previous results in detail.

1.1 Previous works and our contributions

The first systematic study of random walks on Teichmüller space is due to Vadim Kaimanovich and Howard Masur [KM96]. They considered the mapping class group $\mathrm{Mod}(\Sigma)$ as a natural generalization of $\mathrm{SL}(2, \mathbb{Z})$ and compared it with another one, namely, lattices in semi-simple Lie groups. Among many results, they established the escape to infinity of random walks on Teichmüller space $\mathcal{T}(\Sigma)$ and proved that the Thurston boundary of $\mathcal{T}(\Sigma)$ serves as the Poisson boundary for those random walks.

Despite Kaimanovich and Masur's pioneering work, random walks on Teichmüller space are hard to investigate due to the delicate geometric properties of the space. In particular, it is widely known that $\mathcal{T}(\Sigma)$ has different large-scale geometry than negatively curved manifolds. Meanwhile, William J. Harvey suggested a simplicial complex on which $\mathrm{Mod}(\Sigma)$ acts simplicially and named it the complex of curves on Σ [Har79]. A monumental work by Howard Masur and Yair Minsky shows that this curve complex $\mathcal{C}(\Sigma)$ is Gromov hyperbolic [MM99]. Their approach is to carefully investigate the relationship between $\mathcal{T}(\Sigma)$ and $\mathcal{C}(\Sigma)$. For instance, there exists a coarse projection $\pi : \mathcal{T}(\Sigma) \to \mathcal{C}(\Sigma)$ from $\mathcal{T}(\Sigma)$ to $\mathcal{C}(\Sigma)$, and they share the automorphism group $\mathrm{Mod}(\Sigma)$ (up to finite index). Hence, $\mathcal{C}(\Sigma)$ can be chosen as an alternative choice for studying random walks on $\mathrm{Mod}(\Sigma)$.

However, the curve complex has pros and cons: it is Gromov hyperbolic but not locally finite nor proper. The traditional approaches to random walks on Gromov hyperbolic spaces, e.g. [Kai00], begin by finding a μ-invariant measure on the boundary of the space using functional analysis on compact spaces. However, it is hard to implement such a strategy on $\mathcal{C}(\Sigma)$ due to its non-properness. Hence, we need to mix theories for proper, non-hyperbolic $\mathcal{T}(\Sigma)$ and non-proper, hyperbolic $\mathcal{C}(\Sigma)$ in a suitable way. Joseph Maher pursued this strategy to obtain the limiting behavior of $\mathrm{Mod}(\Sigma)$ [Mah11]. Subsequent works by Maher ([Mah10a], [Mah10b], [Mah12]) suggested that many limiting behaviors of random walks on Gromov hyperbolic spaces follow from the escape to infinity and the non-atomness of the hitting measure. These observations led to a comprehensive theory for random walks on (possibly non-proper) Gromov hyperbolic spaces by Maher and Giulio Tiozzo [MT18]. Maher and Tiozzo consider the horofunction compactification of the space, where one can appeal to the traditional arguments using compactness. Then they compare the horoboundary and the Gromov boundary to deduce results in the language of the Gromov boundary.

Later, Yves Benoist and Jean-François Quint suggested the martingale approach to random walks on linear groups and Gromov hyperbolic spaces ([BQ16b], [BQ16a]). Benoist and Quint modified random walks into martingales of cocycles at the cost of bounded errors. Once suitably centered, these martingales are subject to standard limit theorems. By approximating the displacement function with suitable cocycles, we arrive at the limit laws for displacement. The centering process here often requires solving a cohomological equation, hence a boundary structure. The interplay between $\mathcal{T}(\Sigma)$ and $\mathcal{C}(\Sigma)$ comes in again: Maher-Tiozzo's theory furnishes the required concentration inequality on $\mathcal{C}(\Sigma)$, and this can be lifted to $\mathcal{T}(\Sigma)$ to implement Benoist-Quint's strategy. Using this procedure, Camille Horbez deduced the CLT on $\mathcal{T}(\sigma)$ [Hor18]. It also led to the SLLN on $\mathcal{T}(\Sigma)$ in the work of François Dahmani and Camille Horbez [DH18].

Pierre Mathieu and Alessandro Sisto pursued a completely different strategy in [MS20]. Mathieu and Sisto deduced that independent random isometries of Gromov hyperbolic spaces make 'almost aligned' progresses, which add up to the total displacement just as in random walks on $\mathbb{R}$. As a result, the proofs for classical limit laws apply directly. This strategy requires a control of the defect arising from the addition, which we call the deviation inequality. Mathieu and Sisto impose a probabilistic condition (that the random walk has finite exponential moment) and a geometric condition (the acylindricality of the action) to establish strong enough deviation inequality. Recent developments including [BMSS22] and [Gou21] suggest that these conditions can be removed in many cases.

Let us explain the recent result by Sébastien Gouëzel [Gou21]. Traditionally, exponentially decaying (summable, resp.) bounds for the escape to infinity deduced from the exponential (summable, resp.) decay of the harmonic measure of random walks, which required the boundedness of the support (finite second moment, resp.). In contrast, Gouëzel proved the result without any moment condition by devising an ingenious measurable function called the set of pivotal times. The construction of pivotal times gives rise to a partition of random paths and leads to an accurate deviation rate from below.

All of the strategies above rely on the Gromov hyperbolicity of the ambient space. Our contribution here is to remove the reliance on the Gromov hyperbolicity and broaden the theory to more general spaces. Namely, our research initially utilized the hyperbolicity along thick geodesics in $\mathcal{T}(\Sigma)$. This enabled us to argue only on $\mathcal{T}(\Sigma)$ without reference to $\mathcal{C}(\Sigma)$. Later, we generalized this strategy and suggested a general theory using the contracting property of certain isometries. This gives not only a concrete control of the stretch factor of a random mapping class (as opposed to the translation length on the curve complex) but also opens the possibility to study random walks on Teichmüller space, CAT(0) spaces and Outer space in a unified way.

We now elaborate on each result. Theorem A describes SLLNs for displacement and translation length. When μ has finite first moment, the SLLN for displacement is a consequence of the subadditive ergodic theorem and the non-amenability of the mapping class group. When the random walk is on the curve complex and μ has infinite first moment, Joseph Maher and Giulio Tiozzo observed that there exists $K > 0$ such that

$$\liminf_n \frac{1}{n} d(o, \omega_n\, o) > K$$

for almost every $\omega = (\omega_n)_n$. By applying Sébastien Gouëzel's pivotal time construction, we prove that K can be chosen as large as we want.

The SLLN for translation length is considerably trickier than the one for displacement. This is because the translation lengths of mapping classes are not subadditive, whereas the displacements are subadditive. In [MT18], Joseph Maher and Giulio Tiozzo established the SLLN for translation length on $\mathcal{C}(\Sigma)$ when the random walk has bounded support. Their strategy works when the random walk has finite second moment, as François Dahmani and Camille Horbez remarked in [DH18]. Dahmani and Horbez also established the SLLN for translation length on $\mathcal{T}(\Sigma)$ under the finite second moment assumption by lifting the deviation inequality on $\mathcal{C}(\Sigma)$ to $\mathcal{T}(\Sigma)$.

Our contribution here is removing the moment assumption and providing a precise dichotomy between finite and infinite first moment. The first step was made by Hyungryul Baik, Dongryul M. Kim and myself in [BCK]. There, we focused on the stabilizer of a Teichmüller curve in Teichmüller space, which is an isometrically embedded copy of a Poincaré disc. For random walks supported on such a stabilizer (which frequently arises from Thurston's construction), the theory of random walks on Gromov hyperbolic spaces applies immediately. Moreover, since the stabilizer obtained from a typical Thurston's construction is virtually cyclic, we can abelianize the stabilizer and apply the theory of random walk on a Euclidean grid. By doing so, we deduced the linear growth of the translation length in the almost sure sense. The restriction of the subgroup structure was removed in [BCK21] using the so-called *pivoting technique*. The complete version was proved in [Cho21a] by combining Baik-Choi-Kim's idea with Sébastien Gouëzel's pivotal time construction in [Gou21].

Theorem B has been partially observed by many authors. In [MT18], Joseph Maher and Giulio

Tiozzo observed that for random walks ω on Gromov hyperbolic spaces with bounded support,

$$\mathbb{P}[d(o, \omega_n\, o) - \tau(\omega_n) \geq \epsilon n]$$

is exponentially decaying and hence summable for each $\epsilon > 0$. François Dahmani and Camille Horbez further elaborated that the above probability is summable for random walks on Teichmüller space with finite second moment [DH18]. We provide a more delicate estimate on the scale of $\log n$ instead of the linear scale in Theorem B.

Let us now discuss Theorem C. Several independent approaches to the CLT for displacement have been suggested. First, Yves Benoist and Jean-François Quint established the CLT for centerable cocycles on compact G-spaces. This theory serves as a common probabilistic ingredient for Benoist-Quint's CLT for linear groups [BQ16b] and proper Gromov hyperbolic spaces [BQ16a] and Camille Horbez's CLT for Teichmüller space [Hor18]. All of these results are under the optimal finite second moment condition. Meanwhile, an independent approach via the theory of *defected adapted cocycles (DAC)* was suggested by Pierre Mathieu and Alessandro Sisto. Using this theory, they proved a CLT for acylindrical intermediates under the finite exponential moment condition [MS20]. Yet another approach is given by Sébastien Gouëzel in [Gou17]. We provide an independent proof of Horbez's CLT on Teichmüller space by combining the pivoting technique with Mathieu-Sisto's theory of DAC.

Meanwhile, the CLT for translation length has not been discussed previously. We prove this by combining the CLT for displacement with Theorem B. Also proved is a more delicate LIL using the deviation inequalities established in Chapter 5. Finally, the converses of CLTs have not been observed before and are parts of our contributions.

Theorem D for displacement is the main result of Sébastien Gouëzel's recent paper [Gou21]. Also, it is implicitly explained in [Cho21b] that translation length grows linearly outside a set of exponentially decaying probability. We argue here that the growth rate of the translation length can be as close to the escape rate as we want.

The exponential genericity of BGIP elements in non-elementary simple random walks was discussed by Alessandro Sisto in [Sis18] under the assumption that the action of G on X is WPD. We generalize this result to all non-elementary random walks while removing the WPD assumption.

Theorem E originates from a question of Vadim Kaimanovich in [Kai00]. Kaimanovich suggested two criteria, namely the ray approximation and the strip approximation, for a μ-boundary to be maximal. The ray approximation criterion is guaranteed if the random walk exhibits sublinear geodesic tracking. Kaimanovich and Howard Masur modeled the Poisson boundary of Teichmüller space on its Thurston boundary using the strip approximation criterion. Kaimanovich then asked whether random walks on Teichmüller space satisfy the ray approximation criterion also.

This question was partially answered by Moon Duchin [Duc05] by descending to subsequences, and fully answered by Giulio Tiozzo [Tio15]. Tiozzo's approach is general and covers many other interesting settings, including Gromov hyperbolic spaces and CAT(0) spaces. See also [Hor18] for the related deviation inequalities in Teichmüller space and Outer space. Here, our contribution is to obtain a finer rate of tracking, namely, $o(\sqrt{n})$-tracking for random walks with finite first moment.

Meanwhile, sublogarithmic tracking requires stronger moment conditions. Pierre Mathieu and Alessandro Sisto established sublogarithmic tracking of random walks with finite exponential moments on acylindrically hyperbolic groups [MS20]. Joseph Maher and Giulio Tiozzo also obtained the same result for random walks with finite support on weakly hyperbolic groups [MT18]; see also [Led01], [BHM11] and [Sis17] for related results. We recover their results with an independent approach.

Theorem F answers a version of Benson Farb's conjecture in [Far06], asking whether pseudo-Anosovs are generic in the Cayley graph of the mapping class group. Traditionally, counting problems on discrete graphs have been studied using either thermodynamic formalism ([PS98], [Can21]) or the geodesic combing structure ([GTT18], [GTT20b], [GTT20a]). Both approaches require a strong geometric property on the ambient group G, such as Gromov hyperbolicity. Although mapping class groups are known to be automatic, it is not known whether they possess a geodesic combing structure. We suggest a new approach to counting problems and obtain the exponential genericity of pseudo-Anosovs with respect to certain generating sets. In fact, we have better control of the candidates for these generating sets: see Theorem 8.0.1.

1.2 Structure of the article

In Chapter 2 we review the preliminaries. We define the bounded geodesic image property (BGIP) and present lemmata regarding BGIP, whose proofs appear in Chapter 9. We also review the theory of Teichmüller space and random walks. Chapter 3 concerns the alignment and the concatenation lemmata for BGIP axes. These lemmata will be crucial for constructing Schottky sets and the pivoting process.

After these preparations, we define the set of pivotal times and the pivoting process in Chapter 4. We first discuss the pivoting method in the basic setting where only the Schottky choices are modified and the isometries between Schottky slots are unchanged. We also explain two other variations of the original pivoting for later purposes. We then incorporate the pivoting method with random walks and establish the escape to infinity with exponentially decaying error probability.

Chapter 5 deals with the deviation inequalities. After investigating the pivoting process for pairs of independent paths, we define persistent progress that separates the forward and the backward sample paths. By controlling the location of persistent progress, we arrive at the optimal deviation inequalities.

In Chapter 6, we prove CLT and related results for displacement. CLT and LIL directly follow from the deviation inequality, while the converse of the CLT or the nondegeneracy of the limiting Gaussian distribution follow from a more delicate investigation that refers to the pivotal times. Finally, we establish the geodesic tracking of random walks using persistent progress.

In Chapter 7, we establish the limit laws for translation length. We introduce two approaches, one relying on the persistent progress and the other directly using the pivotal times. Using the latter approach, we also establish the exponential bounds for the escape rate from below.

Chapter 8 concernes the exponential genericity of pseudo-Anosovs in a Cayley graph of $\mathrm{Mod}(\Sigma)$. In Chapter 9, we explore other spaces having BGIP isometries. In particular, we prove that fully irreducible outer automorphisms of a free group of finite rank have BGIP in Outer space.

Chapter 2. Preliminaries

2.1 Metric spaces and paths

Let (X, d) be a metric space. For later use, let us employ the notation $d^{sym}(x, y) := d(x, y) + d(y, x)$. We define the *Gromov product* of y and z based at x by

$$(y, z)_x := \frac{1}{2}\big(d(y, x) + d(x, z) - d(y, z)\big).$$

The *diameter* of a set $A \subseteq X$ is defined by

$$\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\},$$

and the *(directed) distances* between sets $A, B \subseteq X$ are defined by

$$d(A, B) := \inf\{d(x, y) : x \in A, y \in B\},$$
$$d^{sym}(A, B) := \inf\{d^{sym}(x, y) : x \in A, y \in B\}.$$

For $R > 0$, the *R-neighborhood* of a set $A \subseteq X$ is defined by

$$\mathscr{N}_R(A) := \{x : d^{sym}(x, A) < R\}.$$

The *Hausdorff distance* between $A, B \subseteq X$ is defined by

$$d_H(A, B) := \inf\{R > 0 : A \subseteq \mathscr{N}_R(B) \text{ and } B \subseteq \mathscr{N}_R(A)\}.$$

An isometry g of X is a bijection from X to X that satisfies $d(gx, gy) = d(x, y)$ for all $x, y \in X$.

Definition 2.1.1 (Geodesics). *A path on X is a map $\gamma : I \to X$ from an interval I or a set of consecutive integers I to X. When I has its minimum and maximum, we call $\gamma(\min I)$ and $\gamma(\max I)$ the* endpoints *of γ and say that γ connects $\gamma(\min I)$ to $\gamma(\max I)$. We also define the* reverse *$\bar{\gamma}$ of a path γ by the composition of γ with the inversion $t \mapsto -t$.*

A subpath *or a* subsegment *of $\gamma : I \to X$ is its restriction $\gamma|_{I \cap J} : I \cap J \to X$ to a nonempty intersection of I with some interval J.*

A path $\gamma : I \to X$ from an interval $I \subseteq \mathbb{R}$ is called a geodesic *if $d(\gamma(s), \gamma(t)) = t - s$ holds for all $s, t \in I$ such that $s < t$.*

*A path $\gamma : I \to X$ is called a K-*quasigeodesic *if*

$$\frac{1}{K}|t - s| - K \le d(\gamma(s), \gamma(t)) \le K|t - s| + K \tag{2.1.1}$$

*holds for all $s, t \in I$ such that $s < t$. If Inequality 2.1.1 holds for all $s, t \in I$, we say that γ is a K-*bi-quasigeodesic.

A metric space X is said to be geodesic *if every ordered pair of points can be connected by a geodesic, i.e., for every $x, y \in X$ there exists a geodesic $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$.*

We will frequently use Inequality 2.1.1 in the following form. For any points p, q on a K-bi-quasigeodesic γ, we have

$$\operatorname{diam}\left(\gamma^{-1}(p) \cup \gamma^{-1}(q)\right) \le K d(p, q) + K^2 \tag{2.1.2}$$

and

$$d(q, p) \le K \operatorname{diam}\left(\gamma^{-1}(p) \cup \gamma^{-1}(q)\right) + K \le K^2 d(p, q) + K^3 + K. \tag{2.1.3}$$

2.2 Contracting sets and bounded geodesic image property

We introduce the notion of contracting sets. Intuitively, metric balls disjoint from these sets are seen as small.

Definition 2.2.1 (contracting sets). *For a subset $A \subseteq X$ of a metric space X and $\epsilon > 0$, we define the* closest point projection *of $x \in X$ to A by*

$$\pi_A(x) := \big\{a \in A : d_X(x,a) = d_X(x,A)\big\}.$$

*A is said to be K-*contracting *if:*

1. *$\pi_A(z) \neq \emptyset$ for all $z \in X$ and*
2. *for all $x, y \in X$ such that $d_X(x,y) \le d_X(x,A) - K$ we have*

$$\operatorname{diam}_X\big(\pi_A(x) \cup \pi_A(y)\big) \le K.$$

*A K-contracting K-quasigeodesic is called a K-*contracting axis*.*

Definition 2.2.2 (Bounded geodesic image property). *A subset $A \subseteq X$ of a geodesic metric space X is said to satisfy the K-*bounded geodesic image property*, or K-BGIP in short, if the following hold:*

1. *for any $z \in X$, $\pi_A(z) \neq \emptyset$;*
2. *for any geodesic η such that $\eta \cap \mathscr{N}_K(A) = \emptyset$, we have $\operatorname{diam}(\pi_A(\eta)) \le K$.*

*A K-quasigeodesic that satisfies K-BGIP is called a K-*BGIP axis*.*

We quote a lemma of Goulnara Arzhantseva, Christopher Cashen and Jing Tao.

Lemma 2.2.3 (Lemma 2.4, [ACT15]). *Let X be a geodesic space. Then a quasigeodesic in X is contracting if and only if it has BGIP.*

Let us now collect some properties of contracting axes.

Lemma 2.2.4 (Continuity of the projection). *Let γ be a K-BGIP axis and $x, y \in X$. Then $\pi_\gamma(\{x,y\})$ has diameter at most $K + d^{sym}(x,y)$.*

Lemma 2.2.5 (Large projections are nearby). *For each $K > 1$ there exists a constant $K' = K'(K)$ that satisfies the following property.*

Let $\gamma : I \to X$ be a K-BGIP axis and $\eta : J \to X$ be a geodesic such that $\operatorname{diam}(\pi_\gamma(\eta)) > K'$. Then for

$$m := \inf \gamma^{-1}\pi_\gamma(\eta), \quad M := \sup \gamma^{-1}\pi_\gamma(\eta),$$

$\gamma([m,M] \cap I)$ is within Hausdorff distance K' from a subsegment of η that contains entire $\eta \cap \mathcal{N}_K(\gamma)$.

Lemma 2.2.6 (Restrictions and nearby sets). *For each $K > 1$ there exists a constant $K' = K'(K)$ such that any subsegment of a K-BGIP axis is a K'-BGIP axis.*

Moreover, if a set A is within Hausdorff distance K from a K-BGIP axis and $\pi_A(z) \neq \emptyset$ for any $z \in X$, then A has K'-BGIP.

Lemma 2.2.7 (No backtracking)**.** *For each $K > 1$ there exists a constant $K' = K'(K)$ that satisfies the following property.*

Let $\gamma : I \to X$ be a K-BGIP axis, $\eta : J \to X$ be a geodesic and $\alpha_i \in J$ be such that $\alpha_1 \le \alpha_2 \le \alpha_3$. Let also $a_1, a_2, a_3 \in I$ be such that $\gamma(a_i) \in \pi_\gamma \eta(\alpha_i)$. Then a_1 and a_3 cannot both belong to $(-\infty, a_2 - K']$ nor $[a_2 + K', +\infty)$.

Lemma 2.2.8 (Fellow traveling)**.** *For each $K > 1$ there exists a constant $K' = K'(K)$ that satisfies the following property.*

Let $\gamma : I \to X$ be a K-quasigeodesic and $\eta_1 : [0, L_1] \to X$, $\eta_2 : [0, L_2] \to X$ be geodesics such that

$$d_H(\gamma, \eta_1), \quad d_H(\gamma, \eta_2) < K, \quad \textit{and} \quad d(\eta_1(0), \eta_2(0)) < K.$$

Then $|L_1 - L_2| < K'$, and η_1 and η_2 K'-fellow travel on the interval $[0, \min\{L_1, L_2\}]$.

These are well-known to experts in this field. Nonetheless, we discuss their proofs for a more general setting in Chapter 9.

Definition 2.2.9 (Isometries with contracting properties)**.** *Let $K > 0$. An isometry g of X is said to be* K-contracting *(K-*BGIP*, resp.) if the orbit $n \in \mathbb{Z} \mapsto g^n o \in X$ is a K-contracting axis (K-BGIP axis, resp.).*

Definition 2.2.10 (Translation length)**.** *For $g \in G$, the* (asymptotic) translation length *of g is defined by*

$$\tau(g) := \liminf_{n\to\infty} \frac{1}{n} d(o, g^n o).$$

An isometry has positive translation length if and only if its orbit $n \mapsto g^n o$ is a quasigeodesic.

Definition 2.2.11 ([BF09, Definition 5.8])**.** *Bi-infinite paths $\kappa = (x_i)_{i\in\mathbb{Z}}$, $\eta = (y_i)_{i\in\mathbb{Z}}$ are said to be* independent *if the map $(n, m) \mapsto d(x_n, y_m)$ is proper, i.e., for any $M > 0$, $\{(n, m) : d(x_n, y_m) < M\}$ is bounded.*

Isometries g, h of X are said to be independent *if their orbits are independent.*

Definition 2.2.12. *A subgroup of* $\mathrm{Isom}(X)$ *is said to be* non-elementary *if it contains two independent BGIP isometries.*

Note that for $a, b \in \mathrm{Isom}(X)$ and $n, m \in \mathbb{Z} \setminus \{0\}$, a^n and b^m are independent BGIP isometries if and only if a and b are so.

2.3 Teichmüller space and the mapping class group

The Teichmüller space $X = \mathcal{T}(\Sigma)$ of a hyperbolic surface Σ is the space of equivalence classes $[(f, \Sigma')]$ of an orientation-preserving homeomorphism $f : \Sigma \to \Sigma'$ from Σ to a hyperbolic surface Σ' (Riemann surface Σ', resp.) of the same type with Σ. Here, (f, Σ_1) and (g, Σ_2) are equivalent if they are homotopic, i.e., if there exists an isometry (conformal mapping, resp.) $i : \Sigma_1 \to \Sigma_2$ such that $i \circ f \simeq g$.

For each pair of points $([(f, \Sigma_1)], [(g, \Sigma_2)])$, there exists a unique representative h in the homotopy class of $g \circ f^{-1} : \Sigma_1 \to \Sigma_2$, called the *Teichmüller mapping*, that attains the minimum of the dilatation K_h. Using this, we define the *Teichmüller metric*

$$d_{\mathcal{T}}\big([(f, \Sigma_1)],\, [(g, \Sigma_2)]\big) := \frac{1}{2} \log K_h = \frac{1}{2} \inf\big\{ \log K_\varphi : \varphi \simeq g \circ f^{-1} \big\}.$$

This metric is Finsler but not Riemannian in general. By modulating the dilatation of the Teichmüller mapping, we get a continuous family of surfaces from $[(f, \Sigma_1)]$ to $[(g, \Sigma_2)]$. This is the unique geodesic connecting two points and is called the *Teichmüller geodesic*.

Let us digress to the discussion on Gromov hyperbolic spaces. In [Gro87], Mikhael Gromov suggested the notion of Gromov hyperbolic spaces in order to embrace negatively curved manifolds, their isometry groups and simplicial trees. Since its definition does not rely on the local structure, one can discuss the Gromov hyperbolicity on Teichmüller space. Nonetheless, $(\mathcal{T}(\Sigma), d_{\mathcal{T}})$ is not Gromov hyperbolic unless Σ is a one-punctured torus or a sphere with at most 4 punctures ([MW95], [Iva02], [Min96a]). Moreover, $(\mathcal{T}(\Sigma), d_{WP})$ is not Gromov hyperbolic unless Σ is a torus with at most two punctures or a sphere with at most 5 punctures [BF06]. Hence, the theory of Gromov hyperbolic spaces does not apply in most cases.

Another generalization of negatively curved manifolds is the category of CAT(0) spaces (see Chapter 9 for details). $(\mathcal{T}(\Sigma), d_{\mathcal{T}})$ is also known to be not CAT(0) unless Σ is a one-punctured torus or a sphere with at most four punctures ([Mas75], [BR18]). Hence, the general theory for CAT(0) spaces also does not apply.

There exists another canonical metric on Teichmüller space, namely, the *Weil-Petersson metric* d_{WP}. This is induced from the inner product on the tangent space to Teichmüller space

$$\langle q_1, q_2 \rangle = \int_X \lambda^2 \bar{q}_1 q_2 \quad (q_1, q_2 \in B(R)),$$

where λ is the hyperbolic metric on the surface R. This metric is known to be CAT(0) (cf. [Tro86], [Wol87]) but is not geodesically complete [Wol75].

The mapping class group $G = \mathrm{Mod}(\Sigma)$ of Σ consists of mapping classes of Σ, i.e., equivalence classes of self-homeomorphisms on Σ up to homotopy. $\mathrm{Mod}(\Sigma)$ naturally acts on $\mathcal{T}(\Sigma)$ as isometries with respect to both metrics. Halsey Royden's theorem [Roy71] and an analogous result by Howard Masur and Michael Wolf [MW02] assert that the isometry groups of $(\mathcal{T}(\Sigma), d_{\mathcal{T}})$ and $(\mathcal{T}(\Sigma), d_{WP})$ are the extended mapping class group $\mathrm{Mod}^{\pm}(\Sigma)$ that contains $\mathrm{Mod}(\Sigma)$ as an index 2 subgroup. Taking the quotient of $\mathcal{T}(\Sigma)$ by the action of $\mathrm{Mod}(\Sigma)$ amounts to forgetting the marking, which yields the moduli space $\mathcal{M}(\Sigma)$ of Σ.

The celebrated Nielsen-Thurston classification asserts that mapping classes are either (i) periodic (finite order), (ii) reducible (i.e., those fixing a multicurve), or (iii) pseudo-Anosov. William Thurston established this classification (see [Thu88] or [FLP79]) by observing the topological dynamics of a mapping class on the so-called *Thurston compactification* $\bar{\mathcal{T}}(\Sigma) = \mathcal{T}(\Sigma) \cup \mathcal{PMF}(\Sigma)$. Lipman Bers later came up with another argument using the Teichmüller geometry as follows [Ber78]. Given a mapping class φ, we define the *minimal translation length*

$$m_{\mathcal{T}}(\varphi) := \inf\{d_{\mathcal{T}}(x, \varphi x) : x \in \mathcal{T}(\Sigma)\}$$

and see whether $m_{\mathcal{T}}(\varphi)$ is achieved at a point or not. We then have the following cases.

1. $m_{\mathcal{T}}(\varphi) = 0$ and is realized at a point $x \in \mathcal{T}(\Sigma)$: then φ belongs to the finite stabilizer of x, and φ is periodic.

2. $m_{\mathcal{T}}(\varphi)$ is not realized: then φ is fixing a multicurve and is said to be reducible.

3. $m_{\mathcal{T}}(\varphi) > 0$ and is realized at a point $x \in \mathcal{T}(\Sigma)$: then φ is said to be pseudo-Anosov and the concatenation of $[\varphi^{i-1}x, \varphi^i x]$ is an infinite precompact geodesic on $\mathcal{T}(\Sigma)$. This is actually the

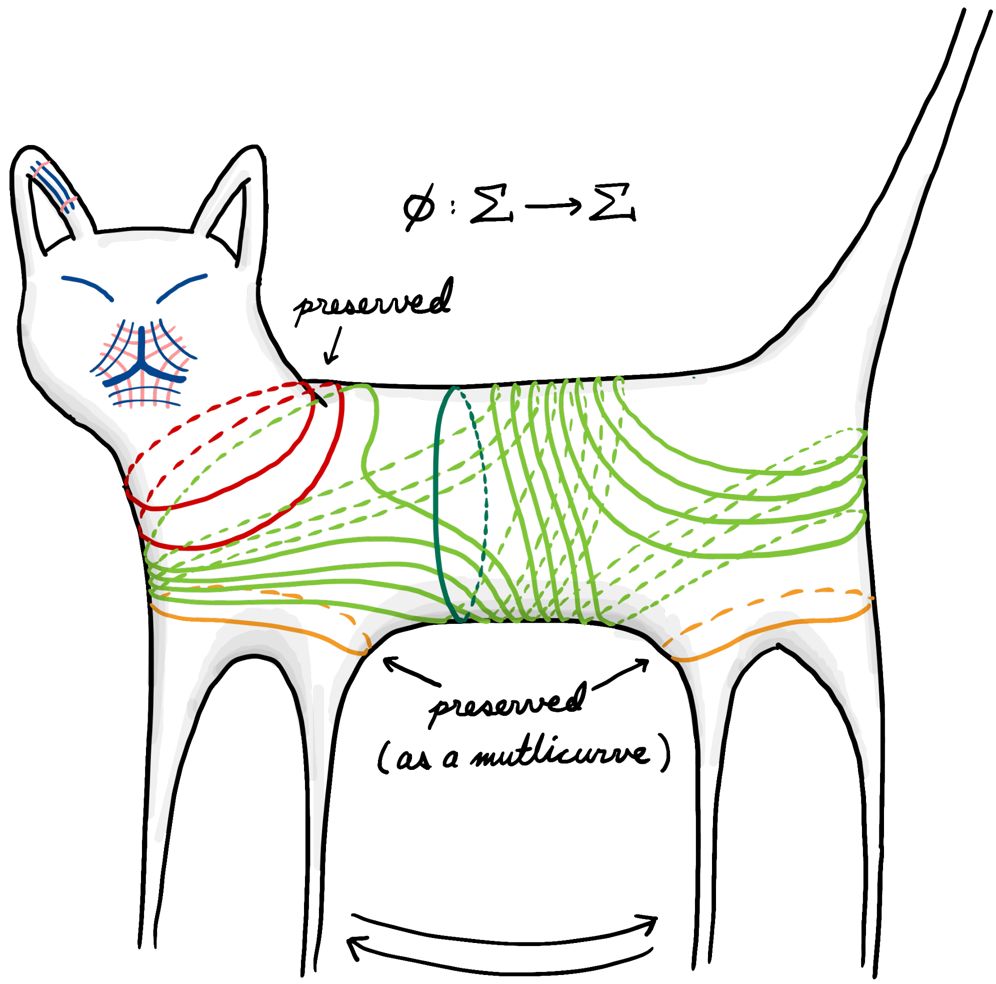

Figure 2.1: A mapping class ϕ on a surface Σ of genus 2 with 5 punctures. ϕ preserves the red curve and the orange multicurve, and its restrictions to two subsurfaces (the head and the body) are pseudo-Anosov. These restrictions preserve mutually transverse measured foliations (blue and pink lines). Also, the green curve on the body subsurface is transformed into a more complicated light green curve by an iteration of ϕ.

unique Teichmüller geodesic that is invariant under the action of φ; we call this the *invariant geodesics* Γ_φ of φ.

Among these, pseudo-Anosovs are considered to have the most interesting dynamics, which have lots of consequences in 3-manifold theory, hyperbolic geometry and the group structure of the mapping class group. For a pseudo-Anosov mapping class φ, its minimal translation length $m_{\mathcal{T}}(\varphi)$ and its (aymptotic) translation length $\tau_X(\varphi)$ are equal. Moreover, there exists a unique representative f and two measured foliations $(\mathcal{F}^\pm, \mu^\pm)$ such that

$$f(\mathcal{F}^+, \mu^+) = (\mathcal{F}^+, \lambda\mu^+), \quad f(\mathcal{F}^-, \mu^-) = \left(\mathcal{F}^-, \frac{1}{\lambda}\mu^-\right)$$

hold for some $\lambda > 1$. We call this λ the *stretch factor* of φ. These foliations can be realized as the horizontal and the vertical foliations of a quadratic differential defined on a point $x \in \mathcal{T}(\Sigma)$ that attains the minimal translation length, and f is the Teichmüller mapping between x and $\varphi \cdot x$. λ is a key dynamical quantity that describes the following phenomena:

1. the maximal dilatation of f equals λ^2, and $\tau_X(\varphi) = \log \lambda$;
2. f is ergodic and Bernoulli with respect to the measure $\mu = \mu^+ \times \mu^-$, and the minimal topological entropy in the equivalence class φ is achieved by f as $\log \lambda$;
3. for any $(f, \Sigma') \in \mathcal{T}(\Sigma)$ and a simple closed curve C on Σ', we have

$$\lim_{n \to +\infty} \sqrt[n]{l_{\Sigma'}([\varphi^n(C)])} = \lambda.$$

Hence, studying the translation lengths of mapping classes on Teichmüller space reveals their dynamical properties on the surface Σ.

The Nielsen-Thurston classification can also be explained with an analogous minimization problem for the Weil-Petersson metric. In [DW03], Georgios Daskalopoulos and Richard Wentworth considered the minimal translation length

$$m_{WP}(\varphi) := \inf \{d_{WP}(x, \varphi x) : x \in \mathcal{T}(\Sigma)\}$$

and obtained the same trichotomy as in the case of the Teichmüller metric. In particular, a pseudo-Anosov mapping class φ has a unique φ-invariant complete Weil-Petersson geodesic. The translation length $\tau_{WP}(\varphi)$ with respect to the Weil-Petersson metric also possesses a geometric meaning. More explicitly, Jeffrey Brock proved in [Bro03] that $\tau_{WP}(\varphi)$ is coarsely related to the hyperbolic volume of the mapping torus made with φ. Hence, investigating the asymptotic behavior of $\tau_{WP}(\varphi)$ leads to some understanding of the volume of a random mapping torus or Heegaard splitting. We refer to Gabriele Viaggi's article [Via21] for another take on this perspective.

Another importance of pseudo-Anosov mapping classes comes from the fact that their invariant geodesics (with respect to either metric) are strongly contracting. Yair Minsky proved that any ϵ-thick geodesics are $K(\epsilon)$-contracting with respect to the Teichmüller metric [Min96b]. For the Weil-Petersson metric, Jason Behrstock established the BGIP of the WP-invariant geodesics of pseudo-Anosov mapping classes using Masur-Minsky's machinery for the curve complex ([Beh06, Theorem 6.5]). For a different approach with the flavor of differential geometry, we refer to [BF09, Proposition 8.1] and [Ham10, Lemma 3.2].

As declared in the introduction, our aim is to establish various limit laws for random walks on $\mathrm{Mod}(\Sigma)$ using the contracting property of pseudo-Anosov mapping classes. Moreover, we will also prove that pseudo-Anosovs predominate in mapping class groups in certain sense. All these results are derived from a single assumption that the random walk sees two independent pseudo-Anosov mapping classes. Hence, we first need a concrete example of a pseudo-Anosov mapping class. For this purpose, William Thurston suggested a recipe (now known as Thurston's construction) to generate pseudo-Anosov mapping classes out of Dehn twists along filling multicurves [Thu88]. Robert Penner later generalized this recipe in [Pen88] to accommodate partial twists along multicurves.

By the work of John McCarthy and Athanase Papadopoulos [MP89], subgroups of $\mathrm{Mod}(\Sigma)$ are either:

- finite;
- reducible, i.e., preserving a multicurve;
- virtually cyclic, or
- non-elementary, i.e., containing two independent pseudo-Anosovs.

If a random walk on $\mathrm{Mod}(\Sigma)$ is supported on a subgroup that falls into the first three categories, then the asymptotic behavior of random walk virtually boils down to random walks on $\{1\}$, $\{\mathbb{Z}\}$ or the mapping class groups of subsurfaces of Σ. This justifies our convention to focus on the non-elementary cases only. Here, the independence of two pseudo-Anosovs means the disjointness of their fixed point set on $\mathcal{PMF}$, but this is equivalent to our Definition 2.2.11.

Let us mention yet another asymmetric metric on Teichmüller space called the *Thurston metric* or *Lipschitz metric* d_L. This metric has been introduced by Thurston [Thu89] from an optimization problem for the lengths of curves on hyperbolic surfaces. Despite its asymmetry, the asymptotic behavior of a random walk on $\mathrm{Mod}(\Sigma)$ with respect to d_L and $d_{\mathcal{T}}$ is essentially the same, because $d_L(x, y)$ and $d_{\mathcal{T}}(x, y)$ differ by a bounded additive error for $x, y \in Go$ [CR07]. This draws a striking contrast with an analogous metric on Outer space that appears in Chapter 9.

2.4 Random walk

For an extensive theory of random walks infinite groups and graphs, we refer to the classic volume by Wolfgang Woess [Woe00].

Let μ be a probability measure on a discrete group G. We consider the *step space* $(G^{\mathbb{Z}}, \mu^{\mathbb{Z}})$, the product space of G equipped with the product measure of μ. Each element $(g_n)_n$ of the step space is called a *step path*, and there is a corresponding *sample path* $(\omega_n)_n$ under the correspondence

$$\omega_n = \begin{cases} g_1 \cdots g_n & n > 0 \\ id & n = 0 \\ g_0^{-1} \cdots g_{n+1}^{-1} & n < 0. \end{cases}$$

This structure constitutes a random walk with transition probability μ. We also introduce the notation $\check{g}_n := g_{-n+1}^{-1}$ and $\check{\omega}_n := \omega_{-n}$.

We define the *support* of μ, denoted by $\mathrm{supp}\,\mu$, as the set of elements in G that are assigned nonzero values of μ. $\langle \mathrm{supp}\,\mu \rangle$ and $\langle\langle\, \mathrm{supp}\,\mu\, \rangle\rangle$ denote the subgroup and the subsemigroup generated by the support of μ, respectively. In other words, we define

$$\begin{aligned} \langle \mathrm{supp}\,\mu \rangle &:= \{g_1 \cdots g_n : n \in \mathbb{Z}_{\geq 0}, g_i \in (\mathrm{supp}\,\mu) \cup (\mathrm{supp}\,\mu)^{-1}\}, \\ \langle\langle\, \mathrm{supp}\,\mu\, \rangle\rangle &:= \{g_1 \cdots g_n : n \in \mathbb{Z}_{\geq 0}, g_i \in \mathrm{supp}\,\mu\}. \end{aligned}$$

We denote by μ^N the product measure of N copies of μ, and by μ^{*N} the N-th convolution measure of μ. A measure μ is said to be *non-elementary* if $\langle\langle\, \mathrm{supp}\,\mu\, \rangle\rangle$ contains two independent contracting isometries. Note that by taking suitable powers if necessary, we may assume that two independent contracting isometries belong to the same $\mathrm{supp}\,\mu^{*N}$ for some $N > 0$. μ is said to be *non-arithmetic* if there exist $N > 0$ and $g, h \in \mathrm{supp}\,\mu^{*N}$ such that $\tau(g) \neq \tau(h)$. The random walk ω generated by μ is said to be admissible (non-elementary or non-arithmetic, resp.) if μ is admissible (non-elementary or non-arithmetic, resp.).

For each $p \geq 0$, we define the p-th moment of the probability measure μ on G by

$$\mathbb{E}_\mu[d(o, go)^p] := \int d(o, go)^p d\mu.$$

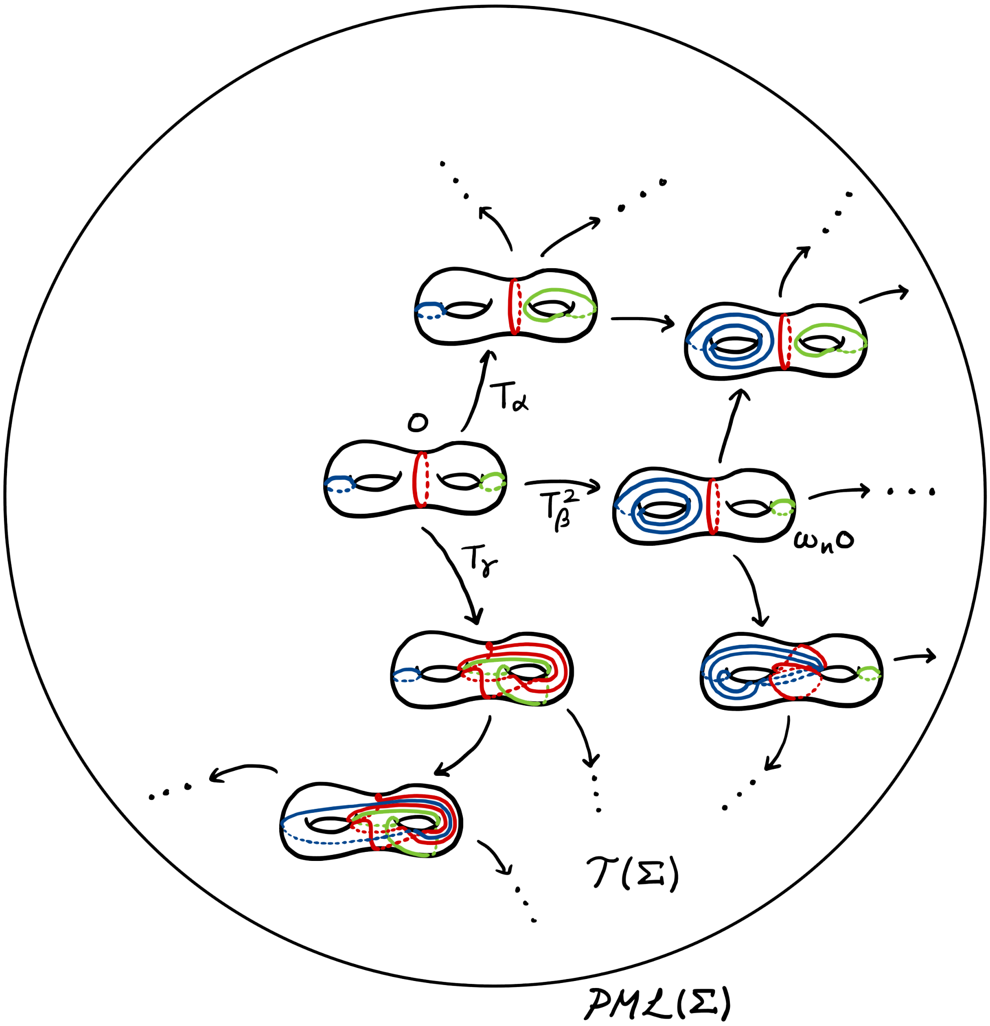

Figure 2.2: A random walk on Teichmüller space.

Chapter 3. Concatenation of BGIP axes

The goal of this chapter is to formulate and prove the following. Let $(\kappa_i)_i$ be a sequence of BGIP axes that begin at x_i and terminate at y_i, respectively. Suppose that consecutive axes are well aligned: κ_i (κ_{i+1}, resp.) projects onto κ_{i+1} (κ_i, resp.) near x_{i+1} (y_i, resp.). Then we have global alignment: κ_i projects onto κ_j near x_j or y_j, depending on whether $i < j$ or $j > i$.

The above statement will be crucial when defining the pivotal times for random walks. In random walks, the desired BGIP axes appear intermittently and the progress in between need not exhibit BGIP. In such a situation, the concatenation lemmata help characterize when $[o, \omega_n o]$ is witnessed by some of the BGIP progresses made at intermediate steps. Note that similar observations were made by Mladen Bestvina and KojiFujiwara to construct nontrivial quasimorphisms for WPD actions on Gromov hyperbolic spaces and CAT(0) spaces (see [BF02], [BF09]).

We note that Wenyuan Yang has previously suggested the prototypes of these concatenation lemmata. In particular, Proposition 3.1.5 and Lemma 3.1.7 were observed earlier in [Yan14, Section 3], and Lemma 3.1.6 follows from [Yan19, Proposition 2.9]. Nonetheless, we include their proofs as applications of Proposition 3.1.4.

3.1 Concatenation lemmata

Definition 3.1.1 (Alignment). *We say that a sequence (κ_1, η) of two paths κ, η is* aligned *if κ projects onto η near the beginning point of η and η projects onto κ near the terminating point of κ.*

More precisely, given paths κ from x to x' and η from y' to y, we say that (κ, η) is C-aligned if

$$\operatorname{diam}(x' \cup \pi_\kappa(\eta)) < C, \quad \operatorname{diam}(y' \cup \pi_\eta(\kappa)) < C.$$

In general, given paths κ_i from x_i to x_i' for each $i = 1, \ldots, n$, we say that $(\kappa_1, \ldots, \kappa_n)$ is C-aligned if

$$\operatorname{diam}(x_i' \cup \pi_{\kappa_i}(\kappa_{i+1})) < C, \quad \operatorname{diam}\left(x_{i+1} \cup \pi_{\kappa_{i+1}}(\kappa_i)\right) < C.$$

hold for $i = 1, \ldots, n-1$.

We can also put points in place of paths in the above definition; in that case, we regard points as degenerate paths that are endpoints of themselves. For example, given $y \in X$ and a path κ connecting x and x', we say that (κ, y) is C-aligned if $\operatorname{diam}(x' \cup \pi_\kappa(y)) < C$.

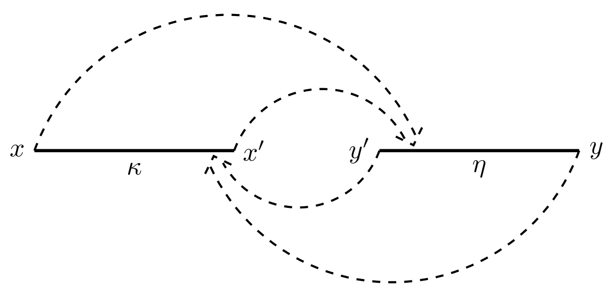

Figure 3.1: Schematics for an aligned sequence of paths.

Note that if sequences $(\kappa_i, \ldots, \kappa_j)$ and $(\kappa_j, \ldots, \kappa_k)$ are C-aligned, then the entire $(\kappa_i, \ldots, \kappa_j, \ldots, \kappa_k)$ is also C-aligned.

Our first lemma states that the projections of endpoints of two BGIP axes onto each other govern the projections of the entire axes.

Lemma 3.1.2. *For each $C > 0$ and $K > 1$, there exists $D = D(K, C) > C$ that satisfies the following property.*

Let κ, η be K-BGIP axes whose beginning and terminating points are x, x' and y', y, respectively. Suppose further that (κ, y') and (x, η) are C-aligned. Then (κ, η) is D-aligned.

One cannot expect similar consequences from the assumption that (κ, y') and (x', η) are aligned: imagine a long and thin isosceles triangle in the hyperbolic plane. Moreover, the assumption that (κ, y) and (x, η) are aligned also cannot guarantee the desired conclusion.

Proof. We let:

- $K_1 = K'(K)$ be as in Lemma 2.2.4;
- $K_2 = K'(K)$ be as in Lemma 2.2.5;
- $K_3 = K'(K)$ be as in Lemma 2.2.7, and
- $D = 16K^{10}(K_1 + K_2 + K_3 + C + 1)$.

If the length L of the domain of κ is smaller than $K(2K + C) + K^2$, then we have

$$\begin{aligned}
\mathrm{diam}(x' \cup \pi_\kappa(\eta)) &\le \mathrm{diam}(\kappa) \le KL + K^2 \le D \\
\mathrm{diam}(y' \cup \pi_\eta(\kappa)) &\le \mathrm{diam}\,(y' \cup \pi_\eta(x)) + \mathrm{diam}\,(\pi_\eta(\kappa)) \\
&\le C + [K_1 + 4\,\mathrm{diam}(\kappa)] \\
&\le C + K_1 + 4KL + 4K^2 \le D.
\end{aligned}$$

The inequality in the third line here is due to Lemma 2.2.4. The desired conclusion follows similarly when the domain of η is shorter than $K(2K + C) + K^2$. Hence, we may assume that the domains of κ, η are longer than $K(2K + C) + K^2$.

The desired conclusion will follow once we show that x' projects onto η near y' and y projects onto κ near x'. More precisely, we claim that the conclusion follows from the inqualities

$$\mathrm{diam}\,(y' \cup \pi_\eta(x')) < 10K^8(K_1 + K_2 + K_3 + C + 1), \tag{3.1.1}$$

$$\mathrm{diam}\,(x' \cup \pi_\kappa(y)) < 10K^8(K_1 + K_2 + K_3 + C + 1). \tag{3.1.2}$$

For example, suppose that Inequality 3.1.2 holds. Recall also that $\mathrm{diam}(x' \cup \pi_\kappa(y')) < C$ by the assumption. Then Inequality 2.1.2 implies

$$\kappa^{-1}(\pi_\kappa(y')), \kappa^{-1}(\pi_\kappa(y)) > \max(J) - [10K^9(K_1 + K_2 + K_3 + C + 1) + K^2],$$

where J denotes the domain of κ. Now Lemma 2.2.7 implies

$$\kappa^{-1}(\pi_\kappa([y, y'])) > \max(J) - [10K^9(K_1 + K_2 + K_3 + C + 1) + K^2 + K_2].$$

Since κ is a K-bi-quasigeodesic, this implies that $x' \cup \pi_\kappa([y, y'])$ has diameter at most

$$K[10K^9(K_1 + K_2 + K_3 + C + 1) + K^2 + K_2] + K \le 13K^{10}(K_1 + K_2 + K_3 + C + 1).$$

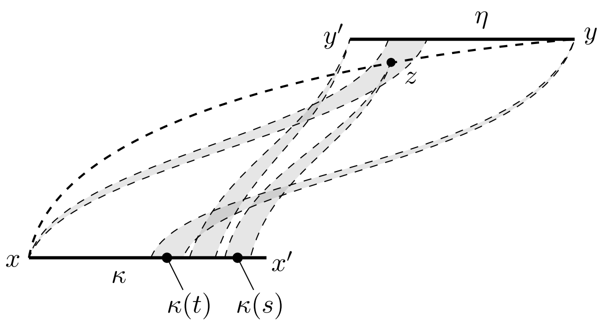

Figure 3.2: Schematics for Lemma 3.1.2

Finally, note that $[y, y']$ and η are within Hausdorff distance K_2 by Lemma 2.2.5 (since η is long enough). Hence, Lemma 2.2.4 implies that $x' \cup \pi_\kappa(\eta)$ has diameter at most $14K^{10}(K_1 + K_2 + K_3 + C + 1)$. Similar argument deduces a bound on the diameter of $y' \cup \pi_\eta(\kappa)$ from the bound on the diameter of $y' \cup \pi_\eta(\{x, x'\})$.

Let us now show that $\pi_\eta(x')$ is near y'. If $\operatorname{diam}(y' \cup \pi_\eta(x')) > 2K + C$, then we have

$$\begin{aligned}\operatorname{diam}(\pi_\eta(x) \cup \pi_\eta(x')) &\geq \operatorname{diam}(y' \cup \pi_\eta(x) \cup \pi_\eta(x')) - \operatorname{diam}(y' \cup \pi_\eta(x)) \\ &\geq (2K + C) - C > K.\end{aligned}$$

Then Lemma 2.2.5 asserts that there exists a point $p \in [x, x']$ such that $d^{sym}(p, y') \leq K_2 + 2C$. Moreover, since κ is long enough, $\operatorname{diam}(\pi_\kappa\{x, x'\}) = \operatorname{diam}(\{x, x'\}) > K$ holds. Again, Lemma 2.2.5 implies that $d_H(\kappa, [x, x']) \leq K_2$ and there exists $q \in \kappa$ such that $d^{sym}(q, p) \leq K_2$. Now for any $q' \in \pi_\kappa(y')$, $d(y', q') \leq d(y', q) \leq 2K_2 + 2C$ and $d(q, y') \leq 2K_2 + 2C$ so $d(q, q') \leq 4K_2 + 4C$. This implies

$$\begin{aligned}d(y', x') &\leq d(y', q') + \operatorname{diam}(x' \cup \pi_\kappa(y')) \leq 2K_2 + 3C, \\ d(x', y') &\leq \operatorname{diam}(x' \cup \pi_\kappa(y')) + d(q', q) + d(q, y') \\ &\leq C + (4K^2(K_2 + C) + K^3 + K) + 2K_2.\end{aligned}$$

Then for any $p' \in \pi_\eta(x')$, we have

$$d(y', p') \leq d(y', x') + d(x', p') \leq d^{sym}(y', x') = 4C + (4K^2(4K_2 + C) + K^3 + K) + 4K_2$$

Since η is a K-bi-quasigeodesic, this implies that $y' \cup \pi_\eta(x')$ has bounded diameter.

We next show that $\pi_\kappa(y)$ is near x'. Note that

$$\begin{aligned}\operatorname{diam}(\pi_\eta([x, y])) &\geq \operatorname{diam}(y' \cup \pi_\eta(x) \cup \pi_\eta(y)) - \operatorname{diam}(y' \cup \pi_\eta(x)) \\ &\geq (2K + C) - C > K.\end{aligned}$$

By K-BGIP of η, we then have a point $z \in [x, y]$ that belongs to $\mathscr{N}_{K_2}(\pi_\eta(x))$. Since $\operatorname{diam}(y' \cup \pi_\eta(x)) < C$, we deduce $d^{sym}(z, y') < K_2 + 2C$. Now Lemma 2.2.4 implies

$$\operatorname{diam}(\pi_\kappa(y') \cup \pi_\kappa(x)) < K_1 + 2K_2 + 4C.$$

Let J be the domain of κ, and $s, t \in J$ be such that $\kappa(s) \in \pi_\kappa(z)$, $\kappa(t) \in \pi_\kappa(y)$. Recall that $\kappa(\min J) = x$ and $\kappa(\max J) = x'$. Since $\operatorname{diam}(x' \cup \pi_\kappa(y')) < C$ and $\operatorname{diam}(\pi_\kappa(y') \cup \pi_\kappa(z)) < K_1 + 2K_2 + 4C$, we have

$$\begin{aligned}\max J - s &\leq K \operatorname{diam}(x' \cup \kappa(s)) + K \\ &\leq K[\operatorname{diam}(x' \cup \pi_\kappa(y')) + \operatorname{diam}(\pi_\kappa(y') \cup \pi_\kappa(z))] + K \\ &\leq K(K_1 + 2K_2 + 5C + 1).\end{aligned}$$

Note here that $\min J$, s, t belong to $\kappa^{-1}\pi_\kappa(x)$, $\kappa^{-1}\pi_\kappa(z)$ and $\kappa^{-1}\pi_\kappa(y)$, respectively. By Lemma 2.2.7, we have either $\min J \geq s - K_3$ or $t \geq s - K_3$. In the former case, we have

$$\begin{aligned} \operatorname{diam}(x' \cup \pi_\kappa(y)) &\leq K|J| + K \\ &= K[(s - \min J) + (\max J - s)] + K \\ &\leq K[K_3 + K(K_1 + 2K_2 + 5C + 1)] + K. \end{aligned}$$

In the latter case, we have

$$\begin{aligned} \operatorname{diam}(x' \cup \pi_\kappa(y)) &\leq \operatorname{diam}(x' \cup \kappa(t)) + \operatorname{diam}(\pi_\kappa(y)) \\ &\leq K(\max J - t) + K + K_1 \\ &\leq K(\max J - s + K_3) + K + K_1 \\ &\leq K(K_3 + K(K_1 + 2K_2 + 5C + 1)) + K + K_1. \end{aligned}$$

□

In the previous lemma, we deduced that the projection of y onto κ favors x' over x since $[x, y]$ has a large projection on η and passes through y'. We can put an arbitrary point p in place of y and expect the same phenomenon, given that the projection of p onto η does not favor y' over y. In other words, p either favors y' over y or favors x' over x. The following lemma captures this:

Lemma 3.1.3 (cf. [BF09, Lemma 5.6]). *For each $C > 0$ and $K > 1$, there exists $D = D(K, C) > C$ that satisfies the following property.*

Let κ, η be K-BGIP axes whose endpoints are x, x' and y', y, respectively. Suppose that (κ, η) is C-aligned. Then for any $p \in X$, $\operatorname{diam}(\pi_\eta(p) \cup y') \geq D$ and $\operatorname{diam}(\pi_\kappa(p) \cup x') \geq D$ cannot happen simultaneously. Moreover, $\operatorname{diam}(\pi_\eta(p) \cup y') \geq D$ implies $d(p, \kappa) \geq d(p, \eta) + K$.

In other words, at least one of the following hold:

- (p, η) is D-aligned;
- (κ, p) is D-aligned.

Moreover, if the first item is not the case, then $d(p, \kappa) \geq d(p, \eta) + K$. Symmetrically, if the second item is not the case, then $d(p, \eta) \geq d(p, \kappa) + K$.

Proof. Let K_1, K_2, K_3 and D be as in the proof of Lemma 3.1.2, and assume $\operatorname{diam}(\pi_\eta(p) \cup y') \geq D$. Since

$$\begin{aligned} \operatorname{diam}(\pi_\eta([p, x])) &\geq \operatorname{diam}(\pi_\eta(p) \cup \pi_\eta(x)) \\ &\geq \operatorname{diam}(\pi_\eta(p) \cup y') - \operatorname{diam}(\pi_\eta(x) \cup y') \\ &\geq D - C > K, \end{aligned}$$

we know that there exists $z \in [p, x]$ such that $d^{sym}(z, \pi_\eta(x)) < K_2$. Then the proof of Lemma 3.1.2 (after putting p in place of y with p) asserts that $\operatorname{diam}(x' \cup \pi_\kappa(p)) \leq D$.

Let us now pick $q' \in \pi_\eta(p)$ such that

$$\max\{d(y', q'), d(q', y')\} \geq \operatorname{diam}(y' \cup \pi_\eta(p)) - 1 \geq D - 1.$$

Since η is a K-bi-quasigeodesic, we deduce $\min\{d(y', q'), d(q', y')\} \geq \frac{1}{K^2}(D-1-K)-K \geq 10(K+K_2+C)$.

Now take any $q \in \kappa$. Note that

$$\begin{aligned} \operatorname{diam}(\pi_\eta([p, q])), \operatorname{diam}(\pi_\eta([p, y'])) &\geq \operatorname{diam}(\pi_\eta(p) \cup y') - \operatorname{diam}(\pi_\eta(q) \cup y') \\ &\geq D - C > K. \end{aligned}$$

Then Lemma 2.2.5 asserts that there exists $z_1 \in [p, q]$ such that $d^{sym}(z_1, \pi_\eta(q)) < K_2$, and $z_2 \in [p, y']$ such that $d^{sym}(q', z_2) < K_2$. We then observe

$$\begin{aligned} d(p,q) &= d(p,z_1) + d(z_1,q) \\ &\geq d(p,y') - d(z_1,y') \\ &\geq d(p,z_2) + d(z_2,y') - d(z_1,y') \\ &\geq d(p,q') + d(q',y') - d^{sym}(q',z_2) - d(z_1,y') \\ &\geq d(p,\eta) + 10(K+K_2+C) - 2K_2 - C \geq d(p,\eta) + K. \end{aligned}$$

Similarly, we can pick $z_1 \in [q, p]$ and $z_2 \in [y', p]$ that satisfy the same conditions. We then conclude $d(q, p) \geq d(q', p) + K \geq d(\eta, p) + K$ for any $q \in \kappa$. Now $d^{sym}(p, q) \geq d^{sym}(p, q') + 2K \geq d^{sym}(p, \eta) + K$ follows. □

We are now ready to prove the main result of this section.

Proposition 3.1.4. *For each $C > 0$ and $K > 1$, there exist $D = D(K, C) > C$ and $L = L(K, C) > C$ that satisfies the following.*

Let J be a nonempty set of consecutive integers, and $p, \{x_i, y_i\}_{i \in J}$ are points in X. For each $i \in J$, let κ_i be a K-BGIP axis connecting x_i and y_i whose domain is longer than L. Suppose also that $(\kappa_i)_{i \in J}$ is C-aligned. Then we have the following:

1. *the statements*

$$(\kappa_i, p) \text{ is } D\text{-aligned}, \quad (p, \kappa_i) \text{ is } D\text{-aligned}$$

cannot hold simultaneously;

2. *the set*

$$\begin{aligned} J_0 &= J_0\Big(p; (\kappa_i)_{i \in J}, D\Big) \\ &:= \left\{ j \in J : \begin{array}{l} (\kappa_i, p) \text{ is } D\text{-aligned for } i \in J \text{ such that } i < j, \\ (p, \kappa_i) \text{ is } D\text{-aligned for } i \in J \text{ such that } i > j \end{array} \right\} \\ &= \left\{ j \in J : \begin{array}{l} \operatorname{diam}(y_i \cup \pi_{\kappa_i}(p)) < D \text{ for } i \in J \text{ such that } i < j, \\ \operatorname{diam}(x_i \cup \pi_{\kappa_i}(p)) < D \text{ for } i \in J \text{ such that } i > j \end{array} \right\} \end{aligned}$$

consists of either a single integer or two consecutive integers;

3. *$\pi_{\cup_i \kappa_i}(p)$ is nonempty and is contained in $\bigcup\{\pi_{\kappa_j}(p) : j \in J_0\}$; and*

4. *(κ_l, κ_m) is D-aligned for any $l, m \in J$ such that $l < m$.*

Proof. Let $D = D(K, C)$ be as in Lemma 3.1.2 and 3.1.3. For the first item, we take large enough L such that $\operatorname{diam}(x_i \cup \pi_{\kappa_i}(p)) < D$ and $\operatorname{diam}(y_i \cup \pi_{\kappa_i}(p)) < D$ cannot happen simultaneously. For example, $L = K(2D + 2K)$ will do. This choices will guarantee the following for each $i \in J$:

$$\begin{aligned} \operatorname{diam}(x_i \cup \pi_{\kappa_i}(p)) < D \Rightarrow \operatorname{diam}(y_i \cup \pi_{\kappa_i}(p)) \geq D, \\ \operatorname{diam}(y_i \cup \pi_{\kappa_i}(p)) < D \Rightarrow \operatorname{diam}(x_i \cup \pi_{\kappa_i}(p)) \geq D. \end{aligned} \tag{3.1.3}$$

This implies that J_0 cannot contain two elements of J that are separated by more than 1. Hence, it suffices to show that J_0 is nonempty.

Suppose, say, there exists m such that (κ_i, p) is D-aligned for all $i \geq m$ (which also subsumes that J is not bounded above). Then Inequality 3.1.3 says that (p, κ_i) is not D-aligned for $i \geq m$, and Lemma

3.1.3 asserts that $d(p,\kappa_{m+n}) < d(p,\kappa_m) - nK$ for all $n \geq 0$; this violates the nonnegativity of the metric. Hence, such m cannot exist and

$$\{i \in J : (\kappa_i, p) \text{ is } D\text{-aligned}\}$$

cannot contain an infinite increasing sequence of consecutive integers. In other words, J is bounded above unless

$$S := \{j \in J : (\kappa_j, p) \text{ is not } D\text{-aligned}\}$$

is nonempty. If S is empty and J is bounded above, then $\max J \in J_0$ clearly holds. Now suppose that S is nonempty and let $j \in S$. Then (κ_j, p) is not D-aligned, which implies that (p, κ_{j+1}) is D-aligned and (κ_{j+1}, p) is not D-aligned if $j+1 \in J$. The induction goes on: (p, κ_i) is D-aligned and (κ_i, p) is not D-aligned for all $i \in J$ such that $i > j$. $(*)$ Note also that for any $k \leq \inf S$, (κ_i, p) is D-aligned for all $i \in J$ such that $i < k$. This implies that $\min S \in J_0$ if exists.

The remaining case is that S is nonempty and $\min S$ does not exists: that means, both J and S is not bounded below. Then $(*)$ implies that (κ_i, p) is not D-aligned for all $i \in J$. By Lemma 3.1.3, we then have $d(p, \kappa_i) < d(p, \kappa_j) - K(j-i)$ for all $i, j \in J$ such that $i < j$. Fixing j and taking small enough i, we obtain a contradiction with the nonnegativity of the metric. Hence, this case does not happen and the second item is established.

We now observe the third and the fourth items. First suppose that J_0 is a singleton $\{j\}$. By definition and Inequality 3.1.3, we have:

$$\begin{aligned} &\operatorname{diam}(x_i \cup \pi_{\kappa_i}(p)) < D, \quad \operatorname{diam}(y_i \cup \pi_{\kappa_i}(p)) > D \quad (i \in J \text{ such that } i > j), \\ &\operatorname{diam}(y_i \cup \pi_{\kappa_i}(p)) < D, \quad \operatorname{diam}(x_i \cup \pi_{\kappa_i}(p)) > D \quad (i \in J \text{ such that } i < j). \end{aligned} \tag{3.1.4}$$

At the moment, if $\operatorname{diam}(y_j \cup \pi_{\kappa_j}(p)) < D$ holds then $j+1$ also belongs to J_0, a contradiction. Hence, we have $\operatorname{diam}(y_i \cup \pi_{\kappa_i}(p)) \geq D$ for $i \in J$ such that $i \geq j$. (Note also that $j \neq \inf J$.) Then Lemma 3.1.3 tells us that $d(y, \kappa_{i+1}) > d(y, \kappa_i)$ for $i \in J \setminus \sup J$ such that $i \geq j$. By a similar reason, $d(y, \kappa_{i-1}) > d(y, \kappa_i)$ for $i \in J \setminus \inf J$ such that $i \leq j$. Hence we conclude $\pi_{\cup_i \kappa_i}(p) = \pi_{\kappa_j}(p)$.

When $J_0 = \{j, j+1\}$, we similarly deduce $\pi_{\cup_i \kappa_i}(p) \subseteq \pi_{\kappa_j}(p) \cup \pi_{\kappa_{j+1}}(p)$.

Let us now take $l, m \in J$ such that $l < m$. We want to show that (κ_l, κ_m) is D-aligned, or equivalently, $\operatorname{diam}(y_l \cup \pi_{\kappa_l}(p)) < D$ for any $p \in \kappa_m$ and $\operatorname{diam}(x_m \cup \pi_{\kappa_m}(p)) < D$ for any $p \in \kappa_l$. Both directly follow from the assumption if $l = m-1$. When $l < m-1$, $J_0 = J_0(p)$ for $p \in \kappa_m$ must contain m because of the second item. Then the first item implies that $l < J_0(p)$ and $\operatorname{diam}(y_l \cup \pi_{\kappa_l}(p)) < D$ as desired. Similarly, $p \in \kappa_l$ implies $J_0(p) < m$ and $\operatorname{diam}(x_m \cup \pi_{\kappa_m}(p)) < D$ as desired. □

Proposition 3.1.5. *For each $C > 0$ and $K > 1$, there exist $E = E(K, C) > C$ and $L = L(K, C) > C$ that satisfy the following. Let $x, y \in X$ and $\kappa_1, \ldots, \kappa_N$ be K-BGIP axes whose domains are longer than L.*

If $(x, \kappa_1, \ldots, \kappa_N, y)$ is C-aligned, then (x, κ_i, y) is E-witnessed for each $i = 1, \ldots, N$. Moreover, $p \in \mathscr{N}_E([x, y])$ and $(x, y)_p < E$ for any $p \in \kappa_i$.

Proof. Proposition 3.1.4 and Lemma 2.2.5 guarantee that the following statements hold for suitable choices of E_1, E_2 and L.

First, (x, κ_1) is E_1-aligned and hence (κ_1, x) is not E_1-aligned. This prevents $J_0\big(x; (\kappa_i)_i, E_1\big)$ from containing elements larger than 1, i.e., $J_0\big(x; (\kappa_i)_i, E_1\big) = \{1\}$. By a similar reason, we have $J_0\big(y; (\kappa_i)_i, E_1\big) = \{N\}$. Consequently we have that (x, κ_i, y) is E_1-aligned for each $i = 1, \ldots, N$. Since κ_i is a long enough K-BGIP axis, there exists a subsegment $[x', y']$ of $[x, y]$ that is within Hausdorff distance E_2 from κ_i. □

We next discuss the contracting of the concatenation of an aligned sequence of contracting axes.

Lemma 3.1.6. *For each $C, M > 0$ and $K > 1$, there exist $K' = K'(K, C, M) > C$ and $L = L(K, C) > C$ that satisfies the following.*

Let J be a nonempty set of consecutive integers and $\{x_i, y_i\}_{i \in J}$ be points in X. For each $i \in J$, let κ_i be a K-BGIP axis connecting x_i and y_i whose domain is longer than L. Suppose that $(\kappa_i)_{i \in J}$ is C-aligned and $d(y_i, x_{i+1}) < M$ for $i \in J \setminus \sup J$. Then $\cup_i \kappa_i$ is a K'-BGIP axis.

Proof. We take $E = E(K, C)$ and $L_1 = L(K, C)$ be as in Proposition 3.1.4, and $L = L_1 + K(2E + K)$.

To show that $\cup_i \kappa_i$ has BGIP, pick $x, y \in X$. If $\mathrm{diam}(\pi_{\kappa_i}(x) \cup \pi_{\kappa_i}(y)) > K$ for some i, then $[x, y]$ passes through the K-neighborhood of κ_i. If not, i.e., if the projections of x and y onto each κ_i are close to each other, we claim that their projections onto $\cup_i \kappa_i$ are also close to each other.

Let $D = D(K, C) > K$ be as in Proposition 3.1.4 and let $j \in J_0 = J_0(x; (\kappa_i)_i, D)$. Then we have the following cases:

1. $\pi_{\kappa_j}(x)$ is distant from both x_j and y_j: then so is $\pi_{\kappa_j}(y)$, and it follows that $J_0(y; (\kappa_i)_i, D) = \{j\}$ also. Hence the projections of x and y onto $\cup_i \kappa_i$ are those onto κ_j, which are close to each other.

2. $\pi_{\kappa_j}(x)$ is close to x_j: then $J_0(x; (\kappa_i)_i, D) \subseteq \{j-1, j\}$, and $\pi_{\kappa_i}(x)$ is far from x_i for $i \neq j$. Since $\pi_{\kappa_i}(x)$ and $\pi_{\kappa_i}(y)$ are close to each other, the same conclusion holds for $\pi_{\kappa_i}(y)$'s. In other words, $J_0(y; (\kappa_i)_i, D) \subseteq \{j-1, j\}$. In this case,

$$\begin{aligned} \mathrm{diam}\left(\pi_{\cup_i \kappa_i}(\{x, y\})\right) &\le \mathrm{diam}\left(\pi_{\kappa_j}(\{x, y\}) \cup \pi_{\kappa_{j-1}}(\{x, y\})\right) \\ &\le \mathrm{diam}\left(\pi_{\kappa_j}(\{x, y\}) \cup x_j\right) + \mathrm{diam}(x_j \cup y_{j-1}) \\ &\quad + \mathrm{diam}\left(y_{j-1} \cup \pi_{\kappa_{j-1}}(x)\right) + \mathrm{diam}\left(\pi_{\kappa_{j-1}}(\{x, y\})\right) \end{aligned}$$

   is bounded. Here, the first and the last term are bounded thanks to the assumption. The second term is at most M, and the third term is also bounded since $j \in J_0(x; (\kappa_i)_i, D)$ so (κ_{j-1}, x) is D-aligned.

3. $\pi_{\kappa_i}(x)$ is close to x_{j+1}: a similar argument works.

We now show that $\cup_i \kappa_i$ is a quasigeodesic. Note that for any $i < j < k$ and $x \in \kappa_i$, $y \in \kappa_j$ and $z \in \kappa_k$, then $(x, \kappa_{i+1}, \ldots, \kappa_j, \ldots, \kappa_{k-1}, z)$ is C-aligned and $(x, z)_y < E$ due to Proposition 3.1.5. In fact, $(x, z)_y$ is also when $x \in \kappa_i$, $z \in \kappa_{i+1}$ and $y = x_{i+1}$. Indeed, (x, κ', z) is C-aligned for the restriction κ' of κ_{i+1} between y and z, so Proposition 3.1.5 tells us that $(x, z)_y < E$ if $d(y, z) > E$; if not $(x, z)_y \le d(y, z)$ is clearly bounded by E.

These bounds on the Gromov products imply the following. For $i < j$, $x \in \kappa_i$ and $y \in \kappa_j$, we have

$$\begin{aligned} d(x, y) &\ge d(x, y_i) + d(x_{i+1}, y_{i+1}) + \ldots + d(x_j, y) - |j - i| E \\ &\ge \frac{1}{2}[d(x, y_i) + d(x_{i+1}, y_{i+1}) + \ldots + d(x_j, y)] - E. \end{aligned}$$

Here, we used the fact that $d(x_k, y_k) \ge \frac{L}{K} - K \ge 2E$ for each k. Since each κ_i is a K-quasigeodesic, we can conclude that $\cup_i \kappa_i$ is also a quasigeodesic. A symmetric argument shows that the reverse of $\cup_i \kappa_i$ is also a quasigeodesic; hence $\cup_i \kappa_i$ is a bi-quasigeodesic. □

The latter part of the previous proof still works even when $d(y_i, x_{i+1})$ is not uniformly bounded, given that the intermediate segments are included. Hence, we obtain the following:

Lemma 3.1.7. *For each $C > 0$ and $K > 1$, there exist $K' = K'(K, C) > C$ and $L = L(K, C) > C$ that satisfy the following.*

Let J be a nonempty set of consecutive integers and $\{x_i, y_i\}_{i \in J}$ be points in X. For each $i \in J$, let κ_i be a K-BGIP axis connecting x_i and y_i whose domains are longer than L. Suppose that $(\kappa_i)_{i \in J}$ is C-aligned. Then the concatenation Γ of $(\ldots, [x_{i-1}, y_{i-1}], [y_{i-1}, x_i], [x_i, y_i], [y_i, x_{i+1}], \ldots)$ is a K'-quasigeodesic.

3.2 Schottky set

Using the previous concatenation lemmata, we can construct arbitrarily many independent directions out of two independent BGIP isometries.

Lemma 3.2.1. *Let $K > 1$ and $\kappa = (x_i)_{i \in \mathbb{Z}}$, $\eta = (y_i)_{i \in \mathbb{Z}}$ be independent K-BGIP axes. Then κ projects onto η small. More precisely, there exists $K' > 0$ such that*

$$\operatorname{diam}(x_0 \cup \pi_\kappa(\eta)) < K'.$$

Moreover, the projection of the forward half of γ onto its backward half is also small. More precisely, K' can be chosen so that

$$\operatorname{diam}\left(x_0 \cup \pi_{\{x_i\}_{i \geq 0}}(\{x_i\}_{i \leq 0})\right) < K'.$$

Proof. Let $K_1 = K'(K)$ be as in Lemma 2.2.5. Let $l \in \mathbb{Z}$ be such that $x_l \in \pi_\kappa(y_0)$. For the first assertion, suppose to the contrary and let $n_i, m_i \in \mathbb{Z}$ be such that $|m_i| \geq i$ and $x_{m_i} \in \pi_\kappa(y_{n_i})$. Note that $|n_i|$ escapes to infinity, as $\cup_{|k| \leq M} \pi_\kappa(y_k)$ is finite for each M. Moreover, since κ, η are K-quasigeodesics, we have $d(x_l, x_{m_i}), d(y_0, y_{n_i}) > K$ for large enough i. For those i's, Lemma 2.2.5 implies that x_{m_i} is contained in the K_1-neighborhood of $[y_0, y_{n_i}]$, which is contained in the K_1-neighborhood of η. In particular, we have $d(x_{m_i}, y_{n_i'}) < 2K_1$ for some $n_i' \in \mathbb{Z}$. This contradicts the independence of κ and η, and we are led to the conclusion.

The second assertion can be deduced in a similar way, using the fact that the forward and the backward half-paths diverge from each other. □

In practice, we employ the restrictions of κ and η on various sets J of consecutive integers. This necessitates the following modification.

Lemma 3.2.2. *Let $K > 1$ and $\kappa = (x_i)_{i \in \mathbb{Z}}$, $\eta = (y_i)_{i \in \mathbb{Z}}$ be independent K-BGIP axes. Then there exists $K' > 0$ such that the following hold:*

1. *$\kappa|_J := (x_i)_{i \in J}$, $\eta|_J := (y_i)_{i \in J}$ are K'-BGIP axes for any set J of consecutive integers;*
2. *for any set J of consecutive integers that contains 0, we have*

$$\operatorname{diam}\left(x_0 \cup \pi_{\kappa|_J}(\eta)\right) < K';$$

3. *for any positive integer M we have*

$$\operatorname{diam}\left(x_0 \cup \pi_{\{x_0, \ldots, x_M\}}(\{x_i : i \leq 0\})\right) < K'$$

Proof. The first item is a part of Lemma 2.2.6; let $K_1 = K'(K)$ be as in Lemma 2.2.6 and $K_2 = K'(K_1)$ be as in Lemma 2.2.5. Let also $l \in \mathbb{Z}$ be such that $y_l \in \pi_\eta(x_0)$ and let $d(x_0, y_l) = D$.

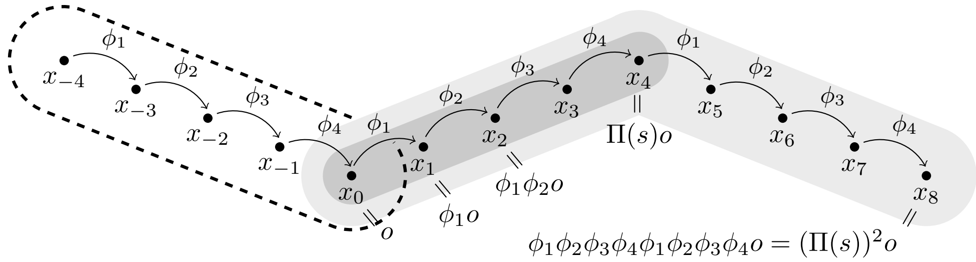

Figure 3.3: Axes associated with a sequence of isometries $s = (\phi_1, \phi_2, \phi_3, \phi_4)$. Points inside the darker shadow constitute $\Gamma(s)$, and those inside the lighter shadow constitute $\Gamma^2(s)$. Points inside the dashed region constitute $\Gamma^{-1}(s)$.

The proof for the second item is almost identical to the proof of the previous lemma. First suppose to the contrary; we take $n_i, m_i \in \mathbb{Z}$ and sets of consecutive integers J_i containing 0 such that $|m_i| \geq i$ and x_{m_i} belongs to the projection of y_{n_i} onto $\kappa|_{J_i}$. Again, for each M we have

$$\begin{aligned}\bigcup_{|i|\leq M,\ 0\in J\subseteq\mathbb{Z}} \pi_{\{x_j : j\in J\}}(y_i) &\subseteq \bigcup_{|i|\leq M} \{x_j : d(y_i, x_j) \leq d(y_i, x_0)\} \\ &\subseteq \bigcup_{|i|\leq M} \{x_j : d(x_0, x_j) \leq d^{sym}(x_0, y_i)\} \\ &\subseteq \{x_j : d(x_0, x_j) \leq D + 2KM + 2K\},\end{aligned}$$

which is a finite set. Hence, $|n_i|$ necessarily escapes to infinity. Moreover, since κ, η are K-quasigeodesics, we have $d(x_0, x_{m_i}), d(y_l, y_{n_i}) > K$ for large enough i. Moreover, $\kappa|_{J_i}$ have the K_1-contracting for all i. Lemma 2.2.5 then asserts that x_{m_i} is within the K_2-neighborhood of $[x_0, y_{n_i}]$, since it has large projection on $\pi|_{J_i}$. Moreover, it is contained in the $(2K_2 + D)$-neighborhood of η. We thus have $d(x_{m_i}, y_{n_i'}) < 3K_2 + D$ for some n_i', which contradicts the independence of κ and η. Hence we are led to the conclusion. Similar trick works for the third item. □

We often construct a path from a sequence of isometries by applying them to the reference point o. Given a sequence $s = (\phi_i)_{i=1}^k$ of isometries of X, we denote the product of its entries $\phi_1 \cdots \phi_k$ by $\Pi(s)$. We also define the reversal of s by $s^{-1} := (\phi_{k-i+1}^{-1})_{i=1}^k$, i.e.,

$$s = (\phi_1, \ldots, \phi_k) \ \Leftrightarrow\ s^{-1} = (\phi_k^{-1}, \ldots, \phi_1^{-1}).$$

Now let

$$x_{nk+i} := \Pi(s)^n \phi_1 \cdots \phi_i o = (\phi_1 \cdots \phi_k)^n \phi_1 \cdots \phi_i o$$

for each $n \in \mathbb{Z}$ and $i = 0, \ldots, k-1$. We let $\Gamma^m(s) := (x_0, x_1, \ldots, x_{mk})$ when $m \geq 0$ and $\Gamma^m(s) := (x_0, x_{-1}, \ldots, x_{mk})$ when $m < 0$. When $m = 1$, we usually omit the superscript and write $\Gamma(s) = (x_0, \ldots, x_k)$. Finally, let $\Gamma^{\pm\infty}(s) = (x_i)_{i\in\mathbb{Z}}$. Note that $\Gamma^{-m}(s) = \Gamma^m(s^{-1})$, and $\Gamma^m(s)$ is a concatenation of $|m|$ translates of $\Gamma(s)$ or its reverse.

We now introduce the notion of Schottky sets. These sets are inspired by the ping-pong dynamics exhibited by classical Schottky Fuchsian groups. Although the idea of Schottky set has appeared in geometric group theory for numerous times, we refer to the versions in [BMSS22] and [Gou21] and adapt them to the current setting.

Definition 3.2.3 (cf. [Gou21, Definition 3.11]). *Let $K > 0$ and $S \subseteq G^M$ be a set of sequences of M isometries. We say that S is K-*Schottky *if the following hold:*

1. $\Gamma^m(s)$ *is a* K*-BGIP axis for all* $s \in S$ *and* $m \in \mathbb{Z}$*;*

2. *for each* $x \in X$*, all element* $s \in S$ *except at most 1 satisfies that* $(x, \Gamma^n(s))$ *is* K*-aligned for all* $n \in \mathbb{Z}$*;*

3. *for each* $x \in X$ *and* $s \in S$*, if* $(x, \Gamma^n(s))$ *is not* K*-aligned for some* $n > 0$ *(*$n < 0$*, resp.) then* $(x, \Gamma^m(s))$ *is* K*-aligned for all* $m \leq 0$ *(*$m \geq 0$*, resp.).*

An intuitive example of a Schottky set is the set S of all sequences of length n in $F_2 = \langle a, b \rangle$ that consists of letters a and b (not involving a^{-1} and b^{-1}). For any infinite ray on F_2, there exists at most 1 element $s \in S$ that matches the direction. Moreover, s and s^{-1} diverge early for any $s \in S$. Note also that the set of the self-concatenations of these sequences also satisfy the same property. This means that we can make the directions made by two sequences in S to diverge early (compared to their lengths). This model will help understand the following proposition.

Proposition 3.2.4 (cf. [Gou21, Proposition 3.12])**.** *For each integer* $N_0 > 0$*, there exists a* K*-Schottky set of cardinality* N_0 *in* $(\operatorname{supp} \mu)^m$ *for some* m *and* K*.*

Proof. Since μ is a non-elementary measure, there exist independent BGIP isometries $a, b \in \langle\langle \operatorname{supp} \mu \rangle\rangle$. By taking suitable powers if necessary, we may assume that $a = \Pi(\alpha)$, $b = \Pi(\beta)$ for some sequences $\alpha, \beta \in (\operatorname{supp} \mu)^N$ for some N. Then $\Gamma^{\pm\infty}(\alpha)$, $\Gamma^{\pm\infty}(\beta)$ are independent contracting axes.

Let:

- $K_1 = K'$ be as in Lemma 3.2.2 for $\Gamma^{\pm\infty}(\alpha), \Gamma^{\pm\infty}(\beta)$;
- $K_2 = D(K_1)$, $L_2 = L'(K_1)$ be as in Proposition 3.1.4;
- $K_3 = K'(K_1)$, $L_3 = L'(K_1)$ be as in Lemma 3.1.6.

Note here that $\Gamma^{\pm\infty}(\alpha)$, $\Gamma^{\pm\infty}(\beta)$ are unchanged after replacing α, β with their self-concatenations. Hence, by self-concatenating α and β if necessary, we may assume that $N > \max(L_2, L_3)$. This choice forces the following: for any $x \in X$, the statements

$$\big(x, \Gamma(\alpha)\big) \text{ is } K_2\text{-aligned}, \quad \big(\Gamma(\alpha), x\big) \text{ is } K_2\text{-aligned}$$

are mutually exclusive. Analogous statements for β are also mutually exclusive. Let us now pick an integer M such that $2^M > N_0$. Since any subset of a Schottky set is again Schottky, we aim to make a Schottky set of cardinality 2^M.

We will consider the set S' of sequences of MN isometries that are concatenations of α's and β's, i.e.,

$$S' := \left\{ (\phi_i)_{i=1}^{MN} \in G^{MN} : (\phi_{N(k-1)+1}, \ldots, \phi_{Nk}) \in \{\alpha, \beta\} \text{ for } k = 1, \ldots, M \right\}.$$

Given $s = (\phi_i)_{i=1}^{MN} \in S'$, we have defined

$$x_{nMN+i}(s) = (\phi_1 \cdots \phi_{MN})^n \phi_1 \cdots \phi_i o$$

for $n \in \mathbb{Z}$ and $i = 0, \ldots, MN - 1$. We temporarily define sub-axes of the main axis $\Gamma(s)$, namely,

$$\Gamma_k(s) := \big(x_{N(k-1)}(s), \ldots, x_{Nk}(s)\big),$$
$$\Gamma_{-k}(s) := \big(x_{-N(k-1)}(s), \ldots, x_{-Nk}(s)\big)$$

for $k = 1, \ldots, M$. Then for each m, $\Gamma^m(s)$ is a concatenation of the translates of $\Gamma(\alpha)$ and $\Gamma(\beta)$. These translates are K_1-BGIP axes whose domains are longer than L_2. Moreover, Lemma 3.2.2 implies that:

$$\left(\overline{\Gamma^{-1}}(\gamma), \Gamma(\gamma')\right) \text{ is } K_1\text{-aligned for } \gamma, \gamma' \in \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\} \text{ such that } \gamma \neq \gamma^{-1}. \tag{3.2.1}$$

Lemma 3.1.6 then implies that $\Gamma^m(s)$ is a K_3-BGIP axis.

We now fix $x \in X$. Let us first consider the condition:

$$\left(x, \Gamma_M(s)\right) \text{ is } K_2\text{-aligned.} \tag{3.2.2}$$

We claim that if an element $s \in S'$ satisfies this condition, then $\pi_{\Gamma^n(s)}(x)$'s are uniformly bounded for $n \geq 0$. For each n, note that $\Gamma^n(s)$ is a concatenation of K_1-BGIP axes

$$(\kappa_i)_{i=1}^{MN} = \left(\Gamma_1(s),\ \ldots,\ \Gamma_M(s),\ \Pi(s)\Gamma_1(s),\ \ldots,\ \Pi(s)\Gamma_M(s),\ \ldots,\ \Pi(s)^{n-1}\Gamma_M(s)\right).$$

Thanks to the result in Display 3.2.1, we can apply Proposition 3.1.4. Note that Condition 3.2.2 implies that $\left(\Gamma_M(s), x\right)$ is not K_2-aligned. This means that $J_0 = J_0\left(x; (\kappa_i)_i, K_2\right)$ and $\{M+1, \ldots, MN\}$ are disjoint. Therefore, $\pi_{\Gamma^n(s)}(x)$ is contained in $\Gamma_1(s) \cup \ldots \cup \Gamma_M(s) = \Gamma(s)$ and

$$\operatorname{diam}\left(\pi_{\Gamma^n(s)}(x) \cup o\right) \leq \operatorname{diam}(\Gamma(s)) \leq K_3 MN + K_3.$$

By a similar reason, the condition

$$\left(x, \Gamma_{-M}(s)\right) \text{ is } K_2\text{-aligned} \tag{3.2.3}$$

implies $\operatorname{diam}\left(\pi_{\Gamma^n(s)}(x) \cup o\right) \leq \operatorname{diam}\left(\Gamma^{-1}(s)\right) \leq K_3 MN + K_3$ for all $n \leq 0$. These can be summarized as follows.

Observation 3.2.5. *If an element $s \in S'$ satisfies Condition 3.2.2 and 3.2.3, then*

$$\operatorname{diam}\left(\pi_{\Gamma^n(s)}(x) \cup o\right) < K_3 MN + K_3$$

holds for all $n \in \mathbb{Z}$.

We now consider the case that an element of S' violates these conditions.

Observation 3.2.6. *If an element $s = (\phi_i)_{i=1}^{MN} \in S'$ violates Condition 3.2.2, then all the other elements $s' = (\phi'_i)_{i=1}^{MN} \in S'$ satisfy Condition 3.2.2.*

To show this, let k be the first index such that $(\phi_{N(k-1)+1}, \ldots, \phi_{Nk})$ and $(\phi'_{N(k-1)+1}, \ldots, \phi'_{Nk})$ differ. By switching the roles of α and β if necessary, we may assume that

$$(\phi_{N(k-1)+1}, \ldots, \phi_{Nk}) = \alpha, \quad (\phi'_{N(k-1)+1}, \ldots, \phi'_{Nk}) = \beta.$$

Let us denote $x_i(s)$ by x_i and $x_i(s')$ by x'_i.

Note that the path

$$\left(x_{MN},\ x_{MN-1},\ \ldots,\ x_{(k-1)N} = x'_{(k-1)N},\ x'_{(k-1)N+1},\ \ldots,\ x'_{kN}\right)$$

is the concatenation of K_1-BGIP axes

$$(\eta_i)_{i=1}^{M-k+2} := \left(\bar{\Gamma}_M(s), \bar{\Gamma}_{M-1}(s), \ldots, \bar{\Gamma}_k(s), \Gamma_k(s')\right).$$

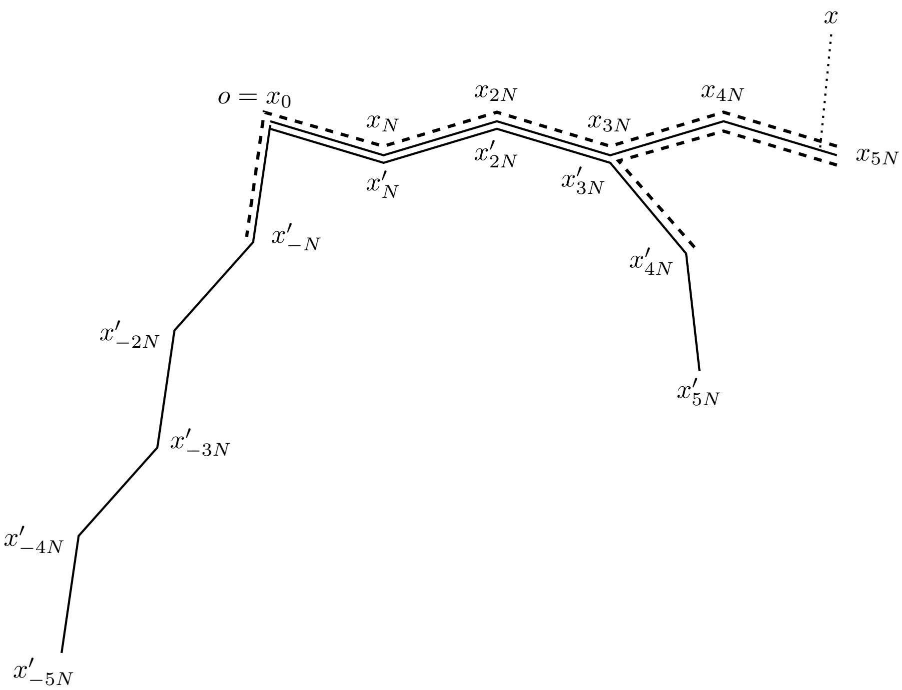

Figure 3.4: Schematics for Lemma 3.2.4. Three solid lines represent $\Gamma(s)$, $\Gamma(s')$ and $\Gamma^{-1}(s')$ in the clockwise order. The upper dashed line represents the concatenation of $\bar{\Gamma}_M(s), \ldots, \bar{\Gamma}_1(s)$ and $\bar{\Gamma}_{-1}(s')$. The lower dashed line represents the concatenation of $\bar{\Gamma}_M, \ldots, \bar{\Gamma}_k(s)$ and $\Gamma_k(s')$.

(See the lower dashed line in Figure 3.4.) Each pair of consecutive axes are of the form $\left(g\bar{\Gamma}^{-1}(\gamma), g\Gamma(\gamma')\right)$ for some $\gamma, \gamma' \in \{\alpha, \beta, \alpha^{-1}, \beta^{-1}\}$ such that $\gamma \neq \gamma'$. Lemma 3.2.2 implies that such pair is K_1-aligned, which allows us to apply Proposition 3.1.4.

In particular, since we are assuming that $\left(\bar{\Gamma}_M(s), x\right)$ is not K_2-aligned, $J_0 = J_0(x; (\eta_i)_i, K_2) = \{1\}$ and $(x, \eta_{M-k+2}) = (x, \Gamma_k(s'))$ is K_2-aligned. We then apply Proposition 3.1.4 to $\Gamma^n(s')$, a concatenation of K_1-BGIP axes

$$(\kappa'_i)_{i=1}^{MN} = \left(\Gamma_1(s'),\ \ldots,\ \Gamma_M(s'),\ \Pi(s')\Gamma_1(s'),\ \ldots,\ \Pi(s')\Gamma_M(s'),\ \ldots,\ \Pi(s')^{n-1}\Gamma_M(s')\right).$$

Then $J'_0 = J_0(x; (\kappa'_i)_i, K_2)$ and $\{k+1, \ldots, MN\}$ are disjoint, which implies Condition 3.2.2 for s' and

$$\pi_{\Gamma^n(s')}(x) \in \Gamma_1(s') \cup \cdots \cup \Gamma_k(s') \subseteq \Gamma(s'),$$
$$\operatorname{diam}\left(\pi_{\Gamma^n(s')}(x) \cup o\right) \leq \operatorname{diam}(\Gamma(s')) \leq K_3 MN + K_3$$

for all $n \geq 0$.

A similar argument leads to the following.

Observation 3.2.7. *If $s \in S'$ violates Condition 3.2.3, then all the other elements in S' satisfy Condition 3.2.3.*

Our next claim concerns the third item.

Observation 3.2.8. *If $s = (\phi_i)_{i=1}^{MN} \in S'$ violates Condition 3.2.2, then all elements $s' = (\phi'_i)_{i=1}^{MN} \in S'$ (including $s' = s$) satisfy Condition 3.2.3.*

To show this, observe that the path

$$\left(x_{MN},\ x_{MN-1},\ \ldots,\ x_0 = o,\ x'_{-1},\ \ldots,\ x'_{-N}\right)$$

is the concatenation of K_0-BGIP axes $\bar{\Gamma}_M(s)$, ..., $\bar{\Gamma}_1(s)$ and $\bar{\Gamma}_{-1}(s')$. (See the upper dashed line in Figure 3.4.) This sequence is again K_1-aligned, even in the case $s = s'$, by Lemma 3.2.2. As before, we can apply Proposition 3.1.4 and deduce that $\pi_{\Gamma_{-1}(s')}(x) \cup o$ has diameter less than K_2. Now Proposition 3.1.4 in turn implies

$$\pi_{\Gamma^{-n}(s')}(x) \in \Gamma_{-1}(s'), \ \operatorname{diam}(\pi_{\Gamma^{-n}(s')}(x) \cup o) \le \operatorname{diam}(\Gamma_{-1}(s')) \le K_3N + K_3$$

for all $n \ge 0$.

An analogous statement follows.

Observation 3.2.9. *If $s = (\phi_i)_{i=1}^{MN} \in S'$ violates Condition 3.2.3, then all elements $s' = (\phi_i')_{i=1}^{MN} \in S'$ (including $s' = s$) satisfy Condition 3.2.2.*

Let us summarize the observations and finish the proof. We take $K = K_3MN + K_3$. The first item was established before. The second item is equivalent to saying that both Condition 3.2.2 and Condition 3.2.3 are satisfied by all but at most 1 element of S'. The third item is equivalent to saying that Condition 3.2.2, 3.2.3 cannot be violated at the same time by any element of S'. We have the following 4 cases.

- Every $s \in S'$ satisfies Condition 3.2.2 and Condition 3.2.3: then clearly the second and the third items hold.
- Some $s \in S'$ violates Condition 3.2.2: then Condition 3.2.2 is satisfied by all the other elements of S' and Condition 3.2.3 is satisfied by all elements of S':
- Some $s \in S'$ violates Condition 3.2.3: then Condition 3.2.3 is satisfied by all the other elements of S' and Condition 3.2.2 is satisfied by all elements of S'.
- Some $s \in S'$ simultaneously violates Condition 3.2.2 and 3.2.3; this case is ruled out by the previous 2 cases.

In all cases, we conclude that the second and the third items hold. □

The following property is immediate.

Lemma 3.2.10. *Let S be a K-Schottky set in G^m for $m > 2K^2$. Then for any $s, s' \in S$, we have*

$$\operatorname{diam}\left(\pi_{\Gamma^{-1}(s')}(\Pi(s)o) \cup o\right) < K, \quad \operatorname{diam}\left(\pi_{\Gamma(s)}(\Pi(s')^{-1}o) \cup o\right) < K. \tag{3.2.4}$$

Proof. For the first inequality, we observe that

$$\operatorname{diam}\left(\pi_{\Gamma(s)}(\Pi(s)o) \cup o\right) = \operatorname{diam}\left(\Pi(s)o \cup o\right) \ge m/K - K > K.$$

Hence, we observe that

$$\operatorname{diam}\left(\pi_{\Gamma^n(s')}(\Pi(s)o) \cup o\right) \le K$$

holds for all n if $s \neq s'$ (Property (2)), and for $n \le 0$ if $s = s'$ (Property (3)); hence the first inequality.

We can analogously deduce the second inequality. □

We will use Schottky sets to guarantee alignments. In order to fully utilize the previous alignment lemmata, it is important to prepare Schottky sets whose elements have sufficiently long domains.

From now on we fix an integer $N_0 > 410$. Let $K_0 := K(N_0)$ be as in Proposition 3.2.4, and

- $K_1 := K'(K_0)$ be as in Lemma 2.2.4,
- $K_2 := K'(K_0)$ be as in Lemma 2.2.5,
- $K_3 := K'(K_0)$ be as in Lemma 2.2.7,
- $D_0 := D(K_0, K_0 + K_1 + K_2 + K_3)$ be as in Lemma 3.1.2 and 3.1.3;
- for $i = 1, 2$, $D_i := D(K_0, D_{i-1})$, $L_i := L(K_0, D_{i-1})$ be as in Lemma 3.1.2, 3.1.3 and Proposition 3.1.4;
- $E_0 := E(K_0, D_2)$, $L_3 := L(C_0, D_2)$ be as in Proposition 3.1.5.

Let us now fix a K_0-Schottky set $S \subseteq (\operatorname{supp}\mu)^{M_0}$ of cardinality at least N_0. Note that the n-self-concatenations of elements of S also comprise a K_0-Schottky set. Hence, we may assume that

$$M_0 > L_1 + L_2 + L_3 + 20K_0(K_0 + E_0). \tag{3.2.5}$$

From now on, K_0-BGIP axes of the form $\Gamma^m(s)$ for $s \in S$ and $m \neq 0$ are called *Schottky axes*.

Chapter 4. Pivotal times and pivoting

4.1 Pivotal times

We adapt Gouëzel's pivotal time construction in [Gou21] to our setting. The original versions of the lemmata here are proved in [Gou21]; see also [Cho21a].

Let $(w_i)_{i=0}^{\infty}$, $(v_i)_{i=1}^{\infty}$ be isometries in G. Now given a sequence

$$s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_n, \beta_n, \gamma_n, \delta_n) \in S^{4n},$$

we first define

$$a_i := \Pi(\alpha_i),\ b_i := \Pi(\beta_i)\ c_i := \Pi(\gamma_i),\ d_i := \Pi(\delta_i). \tag{4.1.1}$$

We then consider isometries that are subwords of

$$w_0 a_1 b_1 v_1 c_1 d_1 w_1 \cdots a_k b_k v_k c_k d_k w_k \cdots .$$

More precisely, we set the initial case $w_{-1,2}^{+} := id$ and define

$$\begin{aligned} w_{i,2}^{-} &:= w_{i-1,2}^{+} w_{i-1}, & w_{i,1}^{-} &:= w_{i,2}^{-} a_i, & w_{i,0}^{-} &:= w_{i,2}^{-} a_i b_i, \\ w_{i,0}^{+} &:= w_{i,2}^{-} a_i b_i v_i, & w_{i,1}^{+} &:= w_{i,2}^{-} a_i b_i v_i c_i, & w_{i,2}^{+} &:= w_{i,2}^{-} a_i b_i v_i c_i d_i \end{aligned}$$

and the translates $y_{i,t}^{\pm} = w_{i,t}^{\pm} o$ of o by them. We also employ notations

$$\begin{aligned} \Upsilon(\alpha_i) &:= w_{i,2}^{-} \Gamma(\alpha_i), & \Upsilon(\beta_i) &:= w_{i,1}^{-} \Gamma(\beta_i), \\ \Upsilon(\gamma_i) &:= w_{i,0}^{+} \Gamma(\gamma_i), & \Upsilon(\delta_i) &:= w_{i,1}^{+} \Gamma(\delta_i). \end{aligned}$$

for simplicity. We will later consider modified versions of a given sequence s such as $\tilde{s} = (\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i, \tilde{\delta}_i)_{i=1}^{n}$ or $\bar{s} = (\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i, \bar{\delta}_i)_{i=1}^{n}$. We also employ notations analogous to the above for these choices, i.e., $\tilde{a}_i$, ..., $\tilde{d}_i$, $\bar{a}_i$, ..., $\bar{d}_i$, $\tilde{w}_{i,j}^{\pm}$, $\bar{w}_{i,j}^{\pm}$ and $\Upsilon(\tilde{\alpha}_i)$, ..., $\Upsilon(\tilde{\delta}_i)$, $\Upsilon(\bar{\alpha}_i)$, ..., $\Upsilon(\bar{\delta}_i)$.

We now define the *set of pivotal times* $P_n = P_n\left(s, (w_i)_{i=0}^{n}, (v_i)_{i=1}^{n}\right)$ and an auxiliary moving point $z_n = z_n\left(s, (w_i)_{i=0}^{n}, (v_i)_{i=1}^{n}\right)$ inductively. First set $P_0 = \emptyset$ and $z_0 = o$. Now given $P_{n-1} \subseteq \{1, \ldots, n-1\}$ and $z_{n-1} \in X$, P_n and z_n are determined as follows.

(A) When $(z_{n-1}, \Upsilon(\alpha_n))$, $\left(\Upsilon(\beta_n), y_{n,1}^{+}\right)$, $\left(y_{n,0}^{-}, \Upsilon(\gamma_n)\right)$ and $\left(\Upsilon(\delta_n), y_{n+1,2}^{-}\right)$ are K_0-aligned, then we set $P_n = P_{n-1} \cup \{n\}$ and $z_n = y_{n,1}^{+}$.

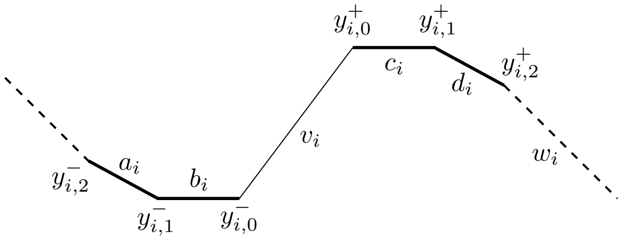

Figure 4.1: $y_{i,k}^{\pm}$ inside a trajectory.

(B) Otherwise, we seek sequences $\{i(1) < \cdots < i(N)\} \subseteq P_{n-1}$ $(N > 1)$ such that

$$\big(\Upsilon(\delta_{i(1)}), \Upsilon(\alpha_{i(2)}), \Upsilon(\beta_{i(2)}), \ldots, \Upsilon(\alpha_{i(N)}), \Upsilon(\beta_{i(N)})\big)$$

is D_0-aligned and $\big(\Upsilon(\beta_{i(N)}), y^-_{n+1,2}\big)$ is K_0-aligned.

If exists, let $\{i(1) < \cdots < i(N)\}$ be such a sequence with maximal $i(1)$; we set $P_n = P_{n-1} \cap \{1, \ldots, i(1)\}$ and $z_n = y^-_{i(N),1}$. If such a sequence does not exist, then we set $P_n = \emptyset$ and $z_n = o$.[1]

One reason for defining P_n is that it records the Schottky axes aligned along $[o, \omega_n\, o]$. More precisely, we have:

Lemma 4.1.1. *Let $P_n = \{i(1) < \ldots < i(m)\}$. Then*

$$\big(o, \Upsilon(\alpha_{i(1)}), \Upsilon(\beta_{i(1)}), \Upsilon(\gamma_{i(1)}), \Upsilon(\delta_{i(1)}), \ldots, \Upsilon(\alpha_{i(m)}), \Upsilon(\beta_{i(m)}), \Upsilon(\gamma_{i(m)}), \Upsilon(\delta_{i(m)}), y^-_{n+1,2}\big)$$

is a subsequence of a D_0-aligned sequence of Schottky axes. In particular, it is D_1-aligned.

This is originally from [Gou21, Lemma 5.3]. We first need the following observation:

Observation 4.1.2. *For any $s \in S^{4n}$ and $1 \le i \le n$, $(\Upsilon(\alpha_i), \Upsilon(\beta_i))$ and $(\Upsilon(\gamma_i), \Upsilon(\delta_i))$ are D_0-aligned.*

Lemma 4.1.3. *Let $l < m$ be consecutive elements in P_k, i.e., $l, m \in P_k$ and $l = \max(P_k \cap \{1, \ldots, m-1\})$. Then there exists a sequence $\{l = i(1) < \ldots < i(M) = m\} \subseteq P_k$ with cardinality $M \ge 2$ such that*

$$\big(\Upsilon(\delta_l), \Upsilon(\alpha_{i(2)}), \Upsilon(\beta_{i(2)}), \ldots, \Upsilon(\alpha_{i(M-1)}), \Upsilon(\beta_{i(M-1)}), \Upsilon(\alpha_m)\big)$$

is D_0-aligned.

Proof. $l, m \in P_n$ implies that $l \in P_l$ and $l, m \in P_m$. In particular, l (m, resp.) is newly chosen at step l (m, resp.) by fulfilling Criterion (A). Hence, $(\Upsilon(\delta_l), y^-_{l+1,2})$ and $(z_{m-1}, \Upsilon(\alpha_m))$ are K_0-aligned $(*)$, and $z_l = y^+_{l,1}$. Moreover, we have $P_m = P_{m-1} \cup \{m\}$ and $l = \max P_{m-1}$.

If $l = m-1$ and m was newly chosen at step $m = l+1$, then $z_{m-1} = z_l = y^+_{l,1}$ holds. Then Lemma 3.1.2 and $(*)$ imply that $(\Upsilon(\delta_l), \Upsilon(\alpha_m))$ is D_0-aligned.

If $l < m-1$, then $l = \max P_{m-1}$ has survived at step $m-1$ by fulfilling Criterion (B); there exist $l = i(1) < \ldots < i(M-1)$ in P_{m-2} (with $M - 1 \ge 2$) such that:

- $\big(\Upsilon(\delta_{i(1)}), \Upsilon(\alpha_{i(2)}), \Upsilon(\beta_{i(2)}), \ldots, \Upsilon(\alpha_{i(M-1)}), \Upsilon(\beta_{i(M-1)})\big)$ is D_0-aligned;
- $\big(\Upsilon(\beta_{i(M-1)}), y^-_{n+1,2}\big)$ is K_0-aligned, and
- z_{m-1} equals $y^-_{i(M-1),1}$, the beginning point of $\Upsilon(\beta_{i(M-1)})$.

We have also observed that $(z_{m-1}, \Upsilon(\alpha_m))$ is K_0-aligned $(*)$. Then Lemma 3.1.2 asserts that $\big(\Upsilon(\beta_{i(M-1)}), \Upsilon(\alpha_m)\big)$ is D_0-aligned as desired. ☐

Proof of Lemma 4.1.1. Considering the previous lemma, it suffices to prove the following:

- $\big(o, \Upsilon(\alpha_{i(1)})\big)$ is K_0-aligned;
- for each $1 \le t \le m$, $\big(\Upsilon(\alpha_{i(t)}), \Upsilon(\beta_{i(t)}), \Upsilon(\gamma_{i(t)}), \Upsilon(\delta_{i(t)})\big)$ is D_0-aligned;

[1] When there are several sequences that realize maximal $i(1)$, we choose the maximum in the lexicographic order on the length of sequences and $i(2)$, $i(3)$, ….

- there exist finitely many Schottky axes $\Upsilon(\delta_{i(m)}) = \Upsilon_1, \ldots, \Upsilon_M$ such that $\big(\Upsilon_1, \ldots, \Upsilon_M, y_{n+1,2}^-\big)$ is D_0-aligned.

Note that for each $t = 1, \ldots, m$, $i(t)$ is newly chosen as a pivotal time at step $i(t)$ by fulfilling Criterion (A). In particular, we have that:

- $(\Upsilon(\alpha_n), \Upsilon(\beta_n))$ is D_0-aligned (Observation 4.1.2);
- $(\Upsilon(\beta_n), \Upsilon(\gamma_n))$ is D_0-aligned since $\big(\Upsilon(\beta_n), y_{n,1}^+\big)$ and $\big(y_{n,0}^-, \Upsilon(\gamma_n)\big)$ are K_0-aligned (Lemma 3.1.2), and
- $(\Upsilon(\gamma_n), \Upsilon(\delta_n))$ is D_0-aligned (Observation 4.1.2).

This guarantees the second item.

We also note that $P_{i(1)-1} = \emptyset$. Indeed, any j in $P_{i(1)-1}$ is smaller than $i(1)$ and would have survived in $P_{i(1)}$ (since what happened at step $i(1)$ was an addition of an element, not a deletion). Since $i(1)$ was not deleted at any later step, such j would also not be deleted till the end and should have appeared in P_n. Since $i(1)$ is the earliest pivotal time in P_n, no such j exists. Hence, $z_{i(1)-1} = o$ and Criterion (A) for $i(1)$ leads to the first item.

We now observe how $i(m)$ survived in P_n. If $i(m) = n$, then it was newly chosen at step n by fulfilling Criterion (A). In particular, $(\Upsilon(\delta_n), y_{n+1,2}^-)$ is K_0-aligned as desired.

If $i(m) \neq n$, then it has survived at step n as the last pivotal time by fulfilling Criterion (B). In particular, there exist $\{i(m) = j(1) < \ldots < j(k)\} \subseteq P_{n-1}$ $(k > 1)$ such that

$$(\kappa_i)_{i=1}^{2k-1} = \big(\Upsilon(\delta_{j(1)}), \Upsilon(\alpha_{j(2)}), \Upsilon(\beta_{j(2)}), \ldots, \Upsilon(\alpha_{j(k)}), \Upsilon(\beta_{j(k)})\big)$$

is D_0-aligned and $\big(\Upsilon(\beta_{j(k)}), y_{n+1,2}^-\big)$ is K_0-aligned. □

From now on, let us endow the Schottky set S with the uniform measure and consider the product measure on S^{4n}. In other words, we assume that $\alpha_i, \beta_i, \gamma_i, \delta_i$ are drawn from S independently. We now discuss when a new pivotal time is added to the set of pivotal times; this tells us how to pivot the direction at a pivotal time without affecting the set of pivotal times.

Lemma 4.1.4. *For $1 \leq k \leq n$, $s \in S^{4(k-1)}$, we have*

$$\mathbb{P}\Big(\#P_k(s, \alpha_k, \beta_k, \gamma_k, \delta_k) = \#P_{k-1}(s) + 1\Big) \geq 1 - 4/N_0.$$

Proof. Recall Criterion (A) for $\#P_k = \#P_{k-1} + 1$. We will examine the required conditions one-by-one.

First, the condition

$$\operatorname{diam}\Big(\pi_{\Upsilon(\gamma_k)}(y_{k,0}^-) \cup y_{k,0}^+\Big) = \operatorname{diam}\big(\pi_{\Gamma(\gamma_k)}(v_k^{-1}o) \cup o\big) < K_0 \tag{4.1.2}$$

depends only on γ_k. This holds for at least $(\#S - 1)$ choices in S.

Similarly, the condition

$$\operatorname{diam}\Big(\pi_{\Upsilon(\delta_k)}(y_{k+1,2}^-) \cup y_{k,2}^+\Big) = \operatorname{diam}\big(\pi_{\Gamma^{-1}(\delta_k)}(w_k o) \cup o\big) < K_0 \tag{4.1.3}$$

depends only on δ_k, and holds for at least $(\#S - 1)$ choices in S.

Now fixing the choice of γ_k, the condition

$$\operatorname{diam}\Big(\pi_{\Upsilon(\beta_k)}(y_{k,1}^+) \cup y_{k,0}^-\Big) = \operatorname{diam}\big(\pi_{\Gamma^{-1}(\beta_k)}(v_k c_k o) \cup o\big) < K_0 \tag{4.1.4}$$

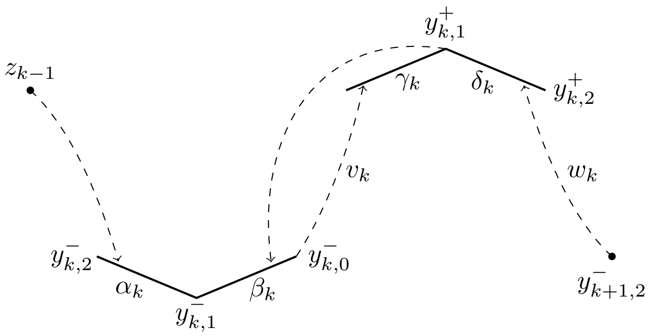

Figure 4.2: Schematics for Criteria 4.1.2, 4.1.3, 4.1.4 and 4.1.5.

depends only on β_k. This holds for at least $(\#S - 1)$ choices in S.

This time, let us fix the choice of $s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}, \delta_{k-1})$; in particular, $w^-_{k,2}$ and z_{k-1} are now determined. Then the condition

$$\operatorname{diam}\left(\pi_{\Upsilon(\alpha_k)}(z_{k-1}) \cup y^-_{k,2}\right) = \operatorname{diam}\left(\pi_{\Gamma(\alpha_k)}\left((w^-_{k,2})^{-1} z_{k-1}\right) \cup o\right) < K_0 \tag{4.1.5}$$

depends on α_k. This holds for at least $(\#S - 1)$ choices of α_k.

In summary, the probability that Criterion (A) holds is at least

$$\frac{\#S-1}{\#S} \cdot \frac{\#S-1}{\#S} \cdot \frac{\#S-1}{\#S} \cdot \frac{\#S-1}{\#S} \geq 1 - \frac{4}{N_0} \qquad \square$$

Given α_1, β_1, γ_1, δ_1, …, α_{k-1}, β_{k-1}, γ_{k-1}, δ_{k-1}, we define the set $\tilde{S}_k$ of triples $(\alpha_k, \beta_k, \gamma_k)$ in S^3 that satisfy Condition 4.1.2, 4.1.4 and 4.1.5. Note that $\tilde{S}_k$ takes up large portion of S^3: in the previous proof we observed that $\#\left[S^3 \setminus \tilde{S}_k\right] \leq 3(\#S)^2$. Moreover, for $(\alpha_k, \beta_k, \gamma_k) \in \tilde{S}_k$, $\{(\alpha_k, \beta'_k, \gamma_k) \in \tilde{S}_k : \beta_k \in S\}$ has at least $\#S - 1$ elements. In addition, $\tilde{S}_k$ is the set of allowed choices for pivoting:

Lemma 4.1.5 ([Gou21, Lemma 5.7])**.** *Let $i \in P_k(s)$ for a choice $s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_n, \beta_n, \gamma_n, \delta_n)$, and $\bar{s}$ be obtained from s by replacing $(\alpha_i, \beta_i, \gamma_i)$ with*

$$(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i) \in \tilde{S}_i(\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}, \delta_{i-1}).$$

Then $P_l(s) = P_l(\bar{s})$ and $\tilde{S}_l(s) = \tilde{S}_l(\bar{s})$ for each $1 \leq l \leq k$.

Proof. Since $\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}, \delta_{i-1}$ are intact, $P_l(s) = P_l(\bar{s})$ and $\tilde{S}'_l(s) = \tilde{S}'_l(\bar{s})$ hold for $l = 0, \ldots, i-1$. At step i, δ_i satisfies Condition 4.1.3 (since $i \in P_k(s)$) and $(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i)$ satisfies Condition 4.1.2, 4.1.4 and 4.1.5. Hence, i is newly added in $P_i(\bar{s})$ and

$$P_i(\bar{s}) = P_{i-1}(\bar{s}) \cup \{i\} = P_{i-1}(s) \cup \{i\} = P_i(s).$$

We also have $\tilde{S}_i(s) = \tilde{S}_i(\bar{s})$ as z_{i-1}, $w^-_{i,2}$ are not affected. Meanwhile, z_i is modified into $\bar{z}_i = \bar{y}^+_{i,1} = gy^+_{i,1} = gz_i$, where $g := w^-_{i,2}\bar{a}_i\bar{b}_i v_i \bar{c}_i (w^-_{i,2} a_i b_i v_i c_i)^{-1}$. More generally, we have

$$\begin{aligned} w^-_{l,t} &= gw^-_{l,t} \quad (t \in \{0,1,2\},\, l > i), \\ w^+_{l,0} &= gw^+_{l,0} \qquad\qquad\quad (l > i), \\ w^+_{l,t} &= gw^+_{l,t} \qquad (t \in \{1,2\}, l \geq i). \end{aligned} \tag{4.1.6}$$

We now claim the following for $i < l \leq k$:

1. If s fulfills Criterion (A) at step l, then so does $\bar{s}$.

2. If not and $\{i(1) < \ldots < i(M)\} \subseteq P_{l-1}(s)$ is the maximal sequence for s in Criterion (B) at step l, then it is also the maximal one for $\bar{s}$ at step l.

3. In both cases, we have $P_l(s) = P_l(\bar{s})$ and $\bar{z}_l = gz_l$.

Assuming the third item for $l-1$: $P_{l-1}(s) = P_{l-1}(\bar{s})$ and $\bar{z}_{l-1} = gz_{l-1}$, Equality 4.1.6 implies the first item. In this case we also deduce $P_l(s) = P_{l-1}(s) \cup \{l\} = P_{l-1}(\bar{s}) \cup \{l\} = P_l(\bar{s})$ and $\bar{z}_l = \bar{y}^+_{l,1} = gy^+_{l,1} = gz_l$, the third item for l.

Furthermore, Equality 4.1.6 implies that a sequence $\{i(1) < \ldots < i(M)\}$ in $P_{l-1}(s) \cap \{i, \ldots, l-1\} = P_{l-1}(\bar{s}) \cap \{i, \ldots, l-1\}$ works for s in Criterion (B) if and only if it works for $\bar{s}$. Note that $i \in P_l(s)$ since $i \in P_k(s)$ and $l \leq k$; hence, such sequences exist and the maximal sequence is chosen among them. Therefore, the maximal sequence $\{i(1) < \ldots < i(M)\}$ for s is also maximal for $\bar{s}$. We then deduce $P_l(s) = P_{l-1}(s) \cap \{1, \ldots, i(1)\} = P_{l-1}(\bar{s}) \cap \{1, \ldots, i(1)\} = P_l(\bar{s})$ and $\bar{z}_l = \bar{y}^-_{i(M),1} = gy^-_{i(M),1} = gz_l$ (noting that $i(M) > i$), the third item for l.

Since we have $\bar{z}_i = gz_i$, induction shows that $P_l(s) = P_l(\bar{s})$ for each $i < l \leq k$. Moreover, Equality 4.1.6 and $\bar{z}_{l-1} = gz_{l-1}$ imply that $\tilde{S}_l(s) = \tilde{S}_l(\bar{s})$. □

Given $1 \leq k \leq n$ and a partial choice $s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_k, \beta_k, \gamma_k, \delta_k)$, we say that $\bar{s} = (\bar{\alpha}_1, \bar{\beta}_1, \bar{\gamma}_1, \bar{\delta}_1, \ldots, \bar{\alpha}_k, \bar{\beta}_k, \bar{\gamma}_k, \bar{\delta}_k)$ is *pivoted from* s if:

- $\delta_j = \bar{\delta}_j$ for all $1 \leq j \leq k$,
- $(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i) \in \tilde{S}_i(s)$ for each $i \in P_k(s)$, and
- $(\alpha_j, \beta_j, \gamma_j) = (\bar{\alpha}_j, \bar{\beta}_j, \bar{\gamma}_j)$ for each $j \in \{1, \ldots, k\} \setminus P_k(s)$.

Lemma 4.1.5 then asserts that being pivoted from each other is an equivalence relation. For each $s \in S^{4k}$, let $\mathcal{E}_k(s)$ be the equivalence class of s. Our central estimation follows:

Lemma 4.1.6 ([Gou21, Lemma 5.8])**.** *For $1 \leq k \leq n$, $j \geq 0$ and $s \in S^{4(k-1)}$, the probability*

$$\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - j \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s),\ (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right)$$

is less than $(4/N_0)^{j+1}$.

Proof. Let us fix $s = (\alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}, \delta_{k-1}) \in S^{4(k-1)}$ and

$$\mathcal{A} := \left\{(\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4 : \#P_k(s, \alpha_k, \beta_k, \gamma_k, \delta_k) = \#P_{k-1}(s) + 1\right\}.$$

Then Lemma 4.1.4 implies that $\mathbb{P}(\mathcal{A} \,|\, S^4) \geq 1 - 4/N_0$. Moreover, for $(\alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}$ we have $P_{k-1}(s) \subseteq P_{k-1}(s) \cup \{k\} = P_k(s, \alpha_k, \beta_k, \gamma_k, \delta_k)$. Hence, $(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k)$ is pivoted from $(s, \alpha_k, \beta_k, \gamma_k, \delta_k)$ for any $\tilde{s} \in \mathcal{E}_{k-1}(s)$. Lemma 4.1.5 then implies that $P_k(\tilde{s}) = P_k(s) = P_{k-1}(s) \cup \{k\} = P_{k-1}(\tilde{s}) \cup \{k\}$, and we have

$$\begin{aligned}&\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(\tilde{s}) \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s), (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right) \\ &\leq 1 - \mathbb{P}(\mathcal{A}|S^4) \leq 4/N_0.\end{aligned}$$

This settles the case $j = 0$.

Now let $j = 1$. The event under discussion becomes void when $\#P_{k-1}(s) < 2$. Excluding such cases, let $l < m$ be the last 2 elements of $P_{k-1}(s)$. For each $\tilde{s} \in \mathcal{E}_{k-1}(s)$ and $A \subseteq S^3$ we define

$$E(\tilde{s}, A) := \left\{ \bar{s} = (\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i, \bar{\delta}_i)_{i=1}^{k-1} : \begin{array}{c} \bar{\alpha}_i = \tilde{\alpha}_i,\ \bar{\gamma}_i = \tilde{\gamma}_i,\ \bar{\delta}_i = \tilde{\delta}_i \text{ for all } i, \\ \bar{\beta}_i = \tilde{\beta}_i \text{ for } i \neq m, \\ (\tilde{\alpha}_m, \bar{\beta}_m, \tilde{\gamma}_m) \in A \end{array} \right\}.$$

In other words, we only modify a single choice of $\tilde{\beta}_m$ in a way that the modified triple at step m belongs to A. Then $\{E(\tilde{s}, \tilde{S}_m(s)) : \tilde{s} \in \mathcal{E}_{k-1}(s)\}$ partitions $\mathcal{E}_{k-1}(s)$ by Lemma 4.1.5. Note that for each $\tilde{s} \in \mathcal{E}_{k-1}(s)$, the size of $E(\tilde{s}, \tilde{S}_m(s))$ is the number of $\bar{\beta}_m \in S$ that satisfies Condition 4.1.4 (with $\tilde{\gamma}_m$ instead of γ_m there); there are at least $\#S - 1$ such choices.

We now fix $(\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4$ and $\tilde{s} = (\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i, \tilde{\delta}_i)_{i=1}^{k-1} \in \mathcal{E}_{k-1}(s)$. Let $\tilde{A} = \tilde{A}(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \subseteq \tilde{S}_m(s)$ be the collection of elements $(\tilde{\alpha}_m, \bar{\beta}_m, \tilde{\gamma}_m)$ in $\tilde{S}_m(s)$ such that $\bar{\beta}_m$ satisfies

$$\begin{aligned} &\operatorname{diam}\left(\pi_{\Gamma^{-1}(\bar{\beta}_m)}((\tilde{w}_{m,0}^{-})^{-1}\tilde{w}_{k-1,2}^{-}a_k b_k v_k c_k d_k o) \cup o\right) \\ &= \operatorname{diam}\left(o \cup \pi_{\Gamma^{-1}(\bar{\beta}_m)}(v_m \tilde{c}_m \tilde{d}_m w_m \cdots \tilde{a}_{k-1}\tilde{b}_{k-1}v_{k-1}\tilde{c}_{k-1}\tilde{d}_{k-1}w_{k-1} \cdot a_k b_k v_k c_k d_k w_k o)\right) < K_0. \end{aligned} \tag{4.1.7}$$

The size of $\tilde{A}$ is the number of $\bar{\beta}_m \in S$ that satisfies Condition 4.1.4 plus Condition 4.1.7; there are at least $\#S - 2$ such choices.

We claim that $\#P_k(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \geq \#P_{k-1}(s) - 1$ for $\bar{s} \in E(\tilde{s}, \tilde{A})$. First, since $l < m$ are consecutive elements in $P_{k-1}(\bar{s})$, Lemma 4.1.3 gives a sequence $\{l = i(1) < \ldots < i(M) = m\} \subseteq P_{k-1}$ such that

$$\left(\Upsilon(\bar{\delta}_{i(1)}), \Upsilon(\bar{\alpha}_{i(2)}), \Upsilon(\bar{\beta}_{i(2)}), \ldots, \Upsilon(\bar{\alpha}_{i(M-1)}), \Upsilon(\bar{\beta}_{i(M-1)}), \Upsilon(\bar{\alpha}_m)\right)$$

is D_0-aligned. Moreover, Observation 4.1.2 and Condition 4.1.7 imply that

$$\left(\Upsilon(\bar{\alpha}_m), \Upsilon(\bar{\beta}_m)\right), \quad \left(\Upsilon(\bar{\beta}_m), \bar{y}_{k+1,2}^{-}\right)$$

are D_0-aligned and K_0-aligned, respectively. In summary, $\{l = i(1) < \cdots < i(M)\} \subseteq P_{k-1}(\bar{s})$ works for $(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k)$ in Criterion (B) at step k. This implies $P_k(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \supseteq P_{k-1}(\bar{s}) \cap \{1, \ldots, l\}$, hence the claim.

As a result, for each $\tilde{s} \in \mathcal{E}_{k-1}(s)$ we have

$$\begin{aligned} &\mathbb{P}\left(\#P_k(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - 1 \,\Big|\, \bar{s} \in E(\tilde{s}, \tilde{S}_m)\right) \\ &\leq \frac{\#\left[E(\tilde{s}, \tilde{S}_m) \setminus E(\tilde{s}, \tilde{A})\right]}{\#E(\tilde{s}, \tilde{S}_m)} \leq \frac{2}{\#S - 1} \leq \frac{3}{N_0}. \end{aligned}$$

Since $E(\tilde{s}, \bar{S}_m)$'s for $\tilde{s} \in \mathcal{E}_{k-1}(s)$ partition $\mathcal{E}_{k-1}(s)$, we deduce

$$\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - 1 \,\Big|\, \tilde{s} \in \mathcal{E}_{k-1}(s)\right) \leq \frac{3}{N_0}.$$

Moreover, the above probability vanishes when $(\alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}$. Since $\mathbb{P}(\mathcal{A} \,|\, S^4) \geq 1 - 4/N_0$, we deduce that

$$\begin{aligned} &\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - 1 \,\Big|\, \tilde{s} \in \mathcal{E}_{k-1}(s), (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right) \\ &\leq \frac{4}{N_0} \cdot \frac{3}{N_0} \leq \left(\frac{4}{N_0}\right)^2. \end{aligned} \tag{4.1.8}$$

Now let $j = 2$. We similarly discuss only for s such that $\#P_{k-1}(s) \geq 3$; let $l' < l < m$ be the last 3 elements. For $(\bar{\alpha}_m, \bar{\beta}_m, \bar{\gamma}_m) \in S^3$ we define

$$s'(\bar{\alpha}_m, \bar{\beta}_m, \bar{\gamma}_m) := (\alpha_1, \beta_1, \gamma_1, \delta_1, \dots, \bar{\alpha}_m, \bar{\beta}_m, \bar{\gamma}_m, \delta_m, \dots, \alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}, \delta_{k-1}).$$

In other words, $s'(\bar{\alpha}_m, \bar{\beta}_m, \bar{\gamma}_m)$ is obtained from s by replacing α_m with $\bar{\alpha}_m$, β_m with $\bar{\beta}_m$ and γ_m with $\bar{\gamma}_m$. We then define

$$\mathcal{A}_1 := \left\{ \begin{pmatrix} \bar{\alpha}_m, \bar{\beta}_m, \bar{\gamma}_m, \\ \alpha_k, \beta_k, \gamma_k, \delta_k \end{pmatrix} \in \tilde{S}_m(s) \times S^4 : \#P_k\left(s'(\bar{\alpha}_m, \bar{\beta}_m, \bar{\gamma}_m), \alpha_k, \beta_k, \gamma_k, \delta_k\right) \geq \#P_{k-1}(s) - 1 \right\}.$$

Equivalently, we are requiring

$$P_{k-1}(s) \cap \{1, \dots, l\} \subseteq P_k\left(s', \alpha_k, \beta_k, \gamma_k, \delta_k\right).$$

(This equivalence relies on the fact $P_{k-1}(s') = P_{k-1}(s)$ due to Lemma 4.1.5.)

Observation 4.1.7. *Let*

$$\tilde{s} = (\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i, \tilde{\delta}_i)_{i=1}^{k-1} \in \mathcal{E}_{k-1}(s), \quad (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4.$$

Then $(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}_1$ *if and only if* $\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \geq \#P_{k-1}(s) - 1$.

To see this, suppose first that $(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}_1$. Then $(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k)$ is pivoted from $\left(s'(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m), \alpha_k, \beta_k, \gamma_k, \delta_k\right)$, as the former choice differs from the latter choice only at entries $(\alpha_i, \beta_i, \gamma_i)$'s for $i \in P_{k-1}(s) \cap \{1, \dots, l\} \subseteq P_k\left(s', \alpha_k, \beta_k, \gamma_k, \delta_k\right)$. Lemma 4.1.5 then implies that

$$P_{k-1}(s) \cap \{1, \dots, l\} \subseteq P_k(s', \alpha_k, \beta_k, \gamma_k, \delta_k) = P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k)$$

and $\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \geq \#P_{k-1}(s) - 1$.

Conversely, suppose $\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \geq \#P_{k-1}(s) - 1$. This amounts to saying

$$P_{k-1}(s) \cap \{1, \dots, l\} \subseteq P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k).$$

Then $\left(s'(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m), \alpha_k, \beta_k, \gamma_k, \delta_k\right)$ is pivoted from $(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k)$, as the former choice differs from the latter choice only at entries $(\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i)$'s for $i \in P_{k-1}(s) \cap \{1, \dots, l\} \subseteq P_k\left(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k\right)$. Lemma 4.1.5 then implies that

$$P_{k-1}(s) \cap \{1, \dots, l\} \subseteq P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) = P_k(s', \alpha_k, \beta_k, \gamma_k, \delta_k)$$

and $(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}_1$.

Combining Observation 4.1.7 and Inequality 4.1.8, we deduce

$$\begin{aligned}
&\mathbb{P}(\mathcal{A}_1 \,|\, \tilde{S}'_m(s) \times S^4) \\
&= \mathbb{P}\left((\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}_1 \,|\, \tilde{s} \in \mathcal{E}_{k-1}(s), (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right) \\
&= \mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \geq \#P_{k-1}(s) - 1 \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s), (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right) \\
&\geq 1 - \left(\frac{4}{N_0}\right)^2.
\end{aligned}$$

We now define for $\tilde{s} \in \mathcal{E}_{k-1}(s)$ and $A \subseteq S^3$

$$E_1(\tilde{s}, A) := \left\{ \bar{s} = (\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i, \bar{\delta}_i)_{i=1}^{k-1} : \begin{array}{c} \bar{\alpha}_i = \tilde{\alpha}_i, \ \bar{\gamma}_i = \tilde{\gamma}_i, \ \bar{\delta}_i = \tilde{\delta}_i \text{ for all } i, \\ \bar{\beta}_i = \tilde{\beta}_i \text{ for } i \neq l, \\ (\tilde{\alpha}_l, \bar{\beta}_l, \tilde{\gamma}_l) \in A \end{array} \right\}.$$

Then $\{E_1(\tilde{s}, \tilde{S}_l(s)) : \tilde{s} \in \mathcal{E}_{k-1}(s)\}$ partitions $\mathcal{E}_{k-1}(s)$ by Lemma 4.1.5. Moreover, for each $\tilde{s} \in \mathcal{E}_{k-1}(s)$ we have $\#E(\tilde{s}, \tilde{S}_l(s)) \geq \#S - 1$.

Now fixing $(\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4$ and $\tilde{s} = (\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i, \tilde{\delta}_i)_{i=1}^{k-1} \in \mathcal{E}_{k-1}(s)$, let $\tilde{A}_1 = \tilde{A}_1(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \subseteq \tilde{S}_l(s)$ be the collection of elements $(\tilde{\alpha}_l, \bar{\beta}_l, \tilde{\gamma}_l)$ that satisfies

$$\operatorname{diam}\left(\pi_{\Gamma^{-1}(\bar{\beta}_l)}((\tilde{w}_{l,0}^{-})^{-1}\tilde{w}_{k-1,2}^{-}a_k b_k v_k c_k d_k o) \cup o\right) < K_0. \tag{4.1.9}$$

As before, the size of $\tilde{A}_1$ is at least $\#S - 2$.

We now claim that $\#P_k(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \geq \#P_{k-1}(s) - 2$ for $\bar{s} \in E_1(\tilde{s}, \tilde{A}_1)$. First, since $l' < l$ are consecutive elements in $P_{k-1}(\bar{s})$, Lemma 4.1.3 gives a sequence $\{l' = i(1) < \ldots < i(M) = l\} \subseteq P_{k-1}$ such that

$$\left(\Upsilon(\bar{\delta}_{i(1)}), \Upsilon(\bar{\alpha}_{i(2)}), \Upsilon(\bar{\beta}_{i(2)}), \ldots, \Upsilon(\bar{\alpha}_{i(M-1)}), \Upsilon(\bar{\beta}_{i(M-1)}), \Upsilon(\bar{\alpha}_l)\right)$$

is D_0-aligned. Moreover, Observation 4.1.2 and Condition 4.1.7 imply that

$$\left(\Upsilon(\bar{\alpha}_l), \Upsilon(\bar{\beta}_l)\right), \quad \left(\Upsilon(\bar{\beta}_l), \bar{y}_{k+1,2}^{-}\right)$$

is D_0-aligned and K_0-aligned, respectively. In summary, $\{l' = i(1) < \cdots < i(M)\} \subseteq P_{k-1}(\bar{s})$ works for $(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k)$ in Criterion (B) at step k. This implies $P_k(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \supseteq P_{k-1}(\bar{s}) \cap \{1, \ldots, l'\}$, hence the claim.

As a result, for each $\tilde{s} \in \mathcal{E}_{k-1}(s)$ we have

$$\begin{aligned}
&\mathbb{P}\left(\#P_k(\bar{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - 2 \,\middle|\, \bar{s} \in E_1(\tilde{s}, \tilde{S}_l)\right) \\
&\leq \frac{\#\left[E(\tilde{s}, \tilde{S}_l) \setminus E(\tilde{s}, \tilde{A}_1)\right]}{\#E(\tilde{s}, \tilde{S}_l')} \leq \frac{2}{\#S - 1} \leq \frac{3}{N_0}.
\end{aligned}$$

Moreover, Observation 4.1.7 asserts that the above probability vanishes for $\tilde{s}$ and $(\alpha_k, \beta_k, \gamma_k, \delta_k)$ such that $(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \in \mathcal{A}_1$. Since

$$\begin{aligned}
&\mathbb{P}\left[\bigcup\{E_1(\tilde{s}, \tilde{S}_l) \times (\alpha_k, \beta_k, \gamma_k, \delta_k) : (\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \notin \mathcal{A}_1\} \,\middle|\, \mathcal{E}_{k-1}(s) \times S^4\right] \\
&= \mathbb{P}\left[(\tilde{\alpha}_m, \tilde{\beta}_m, \tilde{\gamma}_m, \alpha_k, \beta_k, \gamma_k, \delta_k) \notin \mathcal{A}_1 \,\middle|\, \tilde{S}_m(s) \times S^4\right] \leq (4/N_0)^2,
\end{aligned}$$

we sum up the conditional probabilities to obtain

$$\begin{aligned}
&\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - 2 \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s), (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right) \\
&\leq \left(\frac{4}{N_0}\right)^2 \times \frac{3}{N_0} \leq \left(\frac{4}{N_0}\right)^3.
\end{aligned} \tag{4.1.10}$$

We repeat this procedure to cover all $j < \#P_{k-1}(s)$. The case $j \geq \#P_{k-1}(s)$ is void. $\square$

Corollary 4.1.8 ([Gou21, Lemma 5.9, Proposition 5.10]). *When $s = (\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^n$ is chosen from S^{4n} with the uniform measure, $\#P_n(s)$ is greater in distribution than the sum of n i.i.d. X_i, whose distribution is given by*

$$\mathbb{P}(X_i = j) = \begin{cases} (N_0 - 4)/N_0 & \text{if } j = 1, \\ (N_0 - 4)4^{-j}/N_0^{-j+1} & \text{if } j < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.11}$$

More generally, the distribution of $\#P_{k+n}(s) - \#P_k(s)$ conditioned on the choices of $(\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^k$ also dominates the sum of n i.i.d. X_i.

Moreover, we have $\mathbb{P}(\#P_n(s) \leq (1 - 10/N_0)n) \leq e^{-Kn}$ for some $K > 0$.

Proof. Let $\{X_i\}_i$ be the family of i.i.d. as in Equation 4.1.11 that is also assumed to be independent from the choice s. Lemma 4.1.4 and Lemma 4.1.6 imply the following: for $0 \le k < n$ and any i,

$$\mathbb{P}\left(\#P_{k+1}(s) \ge i + j \,\middle|\, \#P_k(s) = i\right) \ge \begin{cases} 1 - \frac{4}{N_0} & \text{if } j = 1, \\ 1 - \left(\frac{4}{N_0}\right)^{-j+1} & \text{if } j < 0. \end{cases} \tag{4.1.12}$$

Hence, there exists a nonnegative RV U_k such that $\#P_{k+1} - U_k$ and $\#P_k + X_{k+1}$ have the same distribution.

For each $1 \le k \le n$, we claim that $\mathbb{P}(\#P_k \ge i) \ge \mathbb{P}(X_1 + \cdots + X_k \ge i)$ for each i. For $k = 1$, we have $\#P_{k-1} = 0$ always and the claim follows from Inequality 4.1.12. Given the claim for k, we have

$$\begin{aligned} \mathbb{P}(\#P_{k+1} \ge i) \ge \mathbb{P}(\#P_k + X_{k+1} \ge i) &= \sum_j \mathbb{P}(\#P_k \ge j)\,\mathbb{P}(X_{k+1} = i - j) \\ &\ge \sum_j \mathbb{P}(X_1 + \cdots + X_k \ge j)\,\mathbb{P}(X_{k+1} = i - j) \\ &= \mathbb{P}(X_1 + \cdots + X_k + X_{k+1} \ge i). \end{aligned}$$

The second assertion follows from a similar induction on $\{\#P_{k+l} - \#P_k\}_{l \ge 0}$.

The final assertion holds since X_i's have finite exponential moments and expectation greater than $1 - 9/N_0$. □

4.2 Variations on the pivotal time construction

In this section, we explain two variants of the pivotal times we defined in Section 4.1.

4.2.1 First variation

We fix subsets $S_1, S_2 \subseteq S$ of cardinality at least $N_0/4$, and a subset $A \subseteq G$. We then assume that for each $s_1 \in S_1, s_2 \in S_2$ and $v \in A$, the two sequences

$$\left(v^{-1}o, \Gamma(s_2)\right), \quad \left(v\Pi(s_2)o, \Gamma^{-1}(s_1)\right) \tag{4.2.1}$$

are K_0-aligned.

As in Section 4.1, we consider the subwords of

$$w_0 a_1 b_1 v_1 c_1 d_1 \cdots a_n b_n v_n c_n d_n w_n \cdots$$

and define $w_{i,j}^{\pm}$, $y_{i,j}^{\pm}$ analogously. This time, however, w_i's are chosen from G and *v_i's are chosen from A*. Also, we will not fix the choice of $(v_i)_i$ this time; only $(w_i)_i$ is fixed. Also, α_i, β_i's are chosen from S_1 and γ_i, δ_i's are chosen from S_2. In other words, a choice $s = (\alpha_1, \beta_1, \ldots, \gamma_n, \delta_n)$ is drawn from $(S_1^2 \times S_2^2)^n$.

Given a choice s, we construct the set of pivotal times $P_n = P_n(s, (w_i)_i, (v_i)_i)$ (with an auxiliary moving point z_n) as in Section 4.1. Then all the lemmata in Section 4.1 are intact, except for some probabilistic estimates. For example, in Lemma 4.1.4 we now have

$$\mathbb{P}\left(\#P_k(s, \alpha_k, \beta_k, \gamma_k, \delta_k) = \#P_{k-1}(s) + 1\right) \ge 1 - 16/N_0,$$

since the choices $\alpha_k, \beta_k, \gamma_k, \delta_k$ are drawn from S_1 or S_2, not the entire S. This also affects Lemma 4.1.6 accordingly. Meanwhile, we have the following variant of Lemma 4.1.5:

Lemma 4.2.1. *Let $i \in P_k(s, \mathbf{v})$ for a choice $s = (\alpha_1, \ldots, \delta_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$. If $\mathbf{v}' = (v_1', \ldots, v_n')$ is made from $\mathbf{v}$ by replacing v_i with an element of A, then $P_l(s, \mathbf{v}) = P_l(s, \mathbf{v}')$ and $\tilde{S}_l(s, \mathbf{v}) = \tilde{S}_l(s, \mathbf{v}')$ for each $1 \le l \le k$.*

Proof. Since $v_1, \ldots, v_{i-1}$ are intact, $P_l(s) = P_l(\bar{s})$ and $\tilde{S}_l'(s, \mathbf{v}) = \tilde{S}_l'(s, \mathbf{v}')$ hold for $l = 0, \ldots, i-1$. At step i, δ_i satisfies Condition 4.1.3 and $\bar{\alpha}_i$ satisfies 4.1.5 since $i \in P_k(s, \mathbf{v})$. Moreover, β_i and γ_i still satisfy Condition 4.1.2 and 4.1.4 after changing v_i into any other element in A, since we assumed Condition 4.2.1. Hence, i is newly added in $P_i(s, \mathbf{v}')$ and

$$P_i(s, \mathbf{v}') = P_{i-1}(s, \mathbf{v}') \cup \{i\} = P_{i-1}(s, \mathbf{v}) \cup \{i\} = P_i(s, \mathbf{v}').$$

We also have $\tilde{S}_i(s) = \tilde{S}_i(\bar{s})$ as z_{i-1}, $w_{i,2}^-$ are not affected, and Condition 4.1.2, 4.1.4 holds for all $\beta_i \in S_1$ and $\gamma_i \in S_2$ thanks to Condition 4.2.1.

Meanwhile, z_i is modified into $\bar{z}_i = \bar{y}_{i,1}^+ = g y_{i,1}^+ = g z_i$, where $g := w_{i,2}^- a_i b_i v_i' (w_{i,2}^- a_i b_i v_i)^{-1}$. More generally, we have

$$\begin{aligned} w_{l,t}^- &= g w_{l,t}^- \quad (t \in \{0,1,2\},\, l > i), \\ w_{l,0}^+ &= g w_{l,0}^+ \quad\quad\quad\quad\quad (l > i), \\ w_{l,t}^+ &= g w_{l,t}^+ \quad\quad (t \in \{1,2\}, l \ge i). \end{aligned} \tag{4.2.2}$$

Now the rest of the proof of Lemma 4.1.5 applies here. $\square$

Given a choice $s = (\alpha_1, \ldots, \delta_n) \in (S_1^2 \times S_2^2)^n$ and $\mathbf{v} = (v_i)_{i=1}^n \in A^n$, we say that $(s, \mathbf{v}')$ is *v-pivoted from* $(s, \mathbf{v})$ if $\mathbf{v}'$ differs from $\mathbf{v}$ only at the pivotal times for $(s, \mathbf{v})$. Then Lemma 4.2.1 tells us that being v-pivoted from each other is an equivalence relation that preserves the set of pivotal times.

4.2.2 Second variation

Again, we only fix $(w_i)_i$ and allow $(v_i)_i$ to vary together with $(\alpha_i, \beta_i, \gamma_i, \delta_i)_i$. However, we do not assume conditions on the candidates for β_i, γ_i and v_i's; $\alpha_i, \beta_i, \gamma_i, \delta_i$'s are chosen from S and v_i's are chosen from G.

We employ the same pivot selection rule as in Section 4.1. However, this time, we define the set $\tilde{S}_k'$ of quadruples $(\alpha_k, \beta_k, \gamma_k, v_k)$ in $S^3 \times G$ that satisfy Condition 4.1.2, 4.1.4 and 4.1.5. Then the proof of Lemma 4.1.5 implies the following:

Lemma 4.2.2. *Let $i \in P_k(s)$ for a choice $s = (\alpha_j, \beta_j, \gamma_j, \delta_j, v_j)_{j=1}^k$ and $\bar{s}$ be obtained from s by replacing $(\alpha_i, \beta_i, \gamma_i, v_i)$ with*

$$(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i, \bar{v}_i) \in \tilde{S}_i'(\alpha_j, \beta_j, \gamma_j, v_j)_{j=1}^{i-1}.$$

Then $P_l(s) = P_l(\bar{s})$ and $\tilde{S}_l'(s) = \tilde{S}_l'(\bar{s})$ for any $1 \le l \le k$.

Thanks to this lemma, we can define the following pivoting. Given a choice $s = (\alpha_l, \beta_l, \gamma_l, \delta_l, v_l)_{l=1}^n$, we say that $\bar{s} = (\bar{\alpha}_l, \bar{\beta}_l, \bar{\gamma}_l, \bar{\delta}_l, \bar{v}_l)_{l=1}^n$ is *pivoted from s in the extended sense* if:

- $\delta_j = \bar{\delta}_j$ for all $1 \le j \le n$,
- $(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i, v_i) \in \tilde{S}_i(s)'$ for each $i \in P_n(s)$, and
- $(\alpha_j, \beta_j, \gamma_j, v_j) = (\bar{\alpha}_j, \bar{\beta}_j, \bar{\gamma}_j, \bar{v}_j)$ for each $j \in \{1, \ldots, n\} \setminus P_n(s)$.

Lemma 4.2.2 then asserts that being pivoted from each other is an equivalence relation.

4.3 Pivotal times in random walks

Let μ_S be the uniform measure on S. By taking suitably small α between 0 and 1, we can decompose μ^{4M_0} as

$$\mu^{4M_0} = \alpha \mu_S^4 + (1-\alpha)\nu$$

for some probability measure ν. We then consider:

- Bernoulli RVs ρ_i with $\mathbb{P}(\rho_i = 1) = \alpha$ and $\mathbb{P}(\rho_i = 0) = 1 - \alpha$,
- η_i with the law μ_S^4, and
- ν_i with the law ν,

all independent, and define

$$(g_{4M_0k+1}, \ldots, g_{4M_0k+4M_0}) = \begin{cases} \nu_k & \text{when } \rho_k = 0, \\ \eta_k & \text{when } \rho_k = 1. \end{cases}$$

Then $(g_i)_{i=1}^{\infty}$ has the law μ^{∞}. We now define Ω to be the ambient probability space on which the above RVs are all measurable. We will denote an element of Ω by ω. We also fix

- $\omega_k := g_1 \cdots g_k$,
- $\mathcal{N}(k) := \sum_{i=0}^{k} \rho_i$, i.e., the number of the Schottky slots till k, and
- $\vartheta(i) := \min\{j \geq 0 : \mathcal{N}(j) = i\}$, i.e., the i-th Schottky slot.

For each $\omega \in \Omega$ and $i \geq 1$ we define

$$\begin{aligned}
w_{i-1} &:= g_{4M_0[\vartheta(i-1)+1]+1} \cdots g_{4M_0\,\vartheta(i)}, \\
\alpha_i &:= (g_{4M_0\,\vartheta(i)+1}, \ldots, g_{4M_0\,\vartheta(i)+M_0}), \\
\beta_i &:= (g_{4M_0\,\vartheta(i)+M_0+1}, \ldots, g_{4M_0\,\vartheta(i)+2M_0}), \\
\gamma_i &:= (g_{4M_0\,\vartheta(i)+2M_0+1}, \ldots, g_{4M_0\,\vartheta(i)+3M_0}), \\
\delta_i &:= (g_{4M_0\,\vartheta(i)+3M_0+1}, \ldots, g_{4M_0\,\vartheta(i)+4M_0}).
\end{aligned}$$

In other words, $\eta_{\vartheta(i)}$ corresponds to $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ (with M_0 steps each) and w_i corresponds to the products of intermediate steps of ν_k's in between $\eta_{\vartheta(i-1)}$ and $\eta_{\vartheta(i)}$. As in Section 4.1, we employ the notation $a_i := \Pi(\alpha_i)$, $b_i := \Pi(\delta_i)$ and so on.

In order to represent ω_n for arbitrary n, we set $n' := \lfloor n/4M_0 \rfloor - 1$ and $w^{(n)} := g_{4M_0[\vartheta(\mathcal{N}(n'))+1]+1} \cdots g_n$. We then have

$$\omega_n = w_0 a_1 b_1 c_1 d_1 w_1 \cdots a_{\mathcal{N}(n')} b_{\mathcal{N}(n')} c_{\mathcal{N}(n')} d_{\mathcal{N}(n')} w^{(n)} \tag{4.3.1}$$

and we can bring the discussion in Section 4.1 here (with v_i's set as id). As before, we denote by s the choices of $\alpha_i, \beta_i, \gamma_i, \delta_i$ and define

$$\begin{aligned}
P_1(\omega) &= P_1\left((w_i)_{i=0}^{1}, a_1, b_1, c_1, d_1\right), \\
P_2(\omega) &= P_2\left((w_i)_{i=0}^{2}, (a_i, b_i, c_i, d_i)_{i=1}^{2}\right), \\
&\vdots
\end{aligned} \tag{4.3.2}$$

and

$$P^{(n)}(\omega) = P_{\mathcal{N}(n')}\left(\left(w_0, \ldots, w_{\mathcal{N}(n')-1}, w^{(n)}\right), (a_i, b_i, c_i, d_i)_{i=1}^{\mathcal{N}(n')}\right).$$

Note that $P^{(n)}(s)$ is built using the decomposition in Equation 4.3.1, and its partial sets of pivotal times are $P_1(\omega)$, ..., $P_{\mathcal{N}(n')-1}(\omega)$. We finally define

$$\mathcal{P}_n(\omega) := \left\{4M_0\, \vartheta(i) : i \in P^{(n)}(s)\right\}.$$

Lemma 4.3.1. *Let ω be a non-elementary random walk on G. Then $\mathcal{P}_n(\omega)$ increases linearly outside a set of exponentially decaying probability. More precisely, there exists $K > 0$ such that*

$$\mathbb{P}\left(\#\,\mathcal{P}_m(\omega) - \#\,\mathcal{P}_m(\omega) \le K(m-n)\right) \le \frac{1}{K} e^{-K(m-n)}$$

holds for all $0 \le n \le m$.

Proof. We denote $\lfloor m/4M_0 \rfloor$ by m' and $\lfloor n/4M_0 \rfloor$ by n'. Recall that the first model involves independent RVs $\{\rho_i, \eta_i, \nu_i\}$'s. We first draw choices of $\{\rho_i\}_{i=1}^m$ that determine the values of $\mathcal{N}(n')$ and $\{\vartheta(1), \ldots, \vartheta(\mathcal{N}(n'))\}$. Since ρ_i has uniform exponential moment and uniform positive expectation, $\mathcal{N}(n')$ increases linearly outside a set of exponentially decaying probability. More precisely, there exists K_1 (independent of m, n) such that for any m, n,

$$\mathbb{P}\left(\mathcal{N}(m') - \mathcal{N}(n') \le K_1(m-n)\right) \le \frac{1}{K_1} e^{-K_1(m-n)}. \tag{4.3.3}$$

Let us fix choices of $\{\rho_i\}_{i=1}^m$ that makes $\mathcal{N}(m') - \mathcal{N}(n') > K_1(m-n)$.

We then draw choices of $\{\nu_i\}_{i=1}^m$ that determine the values of $\{w_{i-1}\}_{i=1}^{\mathcal{N}(m')}$, $w^{(n)}$ and $w^{(m)}$. Now the values of $\{\alpha_i, \beta_i, \gamma_i, \delta_i\}_{i=1}^{\mathcal{N}(m')}$ are determined by the values of $\{\eta_{\vartheta(1)}, \ldots, \eta_{\vartheta(\mathcal{N}(m'))}\}$, which follow the law of $\mu_S^{4\mathcal{N}(n')}$. Now Corollary 4.1.8 provides a constant $K_2 > 0$ such that the following holds:

$$\begin{aligned}
&\mathbb{P}\left(\#\,\mathcal{P}_m(\omega) - \#\,\mathcal{P}_n(\omega) \le K_2(m-n)\right) \\
&\le \mathbb{P}\left(\#P^{(m)}(\omega) - \#P_{\mathcal{N}(n')-1}(\omega) \le K_2(m-n) + 1\right) \\
&\le \frac{1}{K_2} e^{-K_2\left(\mathcal{N}(m') - \mathcal{N}(n')\right)} \le \frac{1}{K_2} e^{-K_2 K_1 (m-n)}.
\end{aligned}$$

Here, the first inequality is due to the relationship

$$\#\,\mathcal{P}_n(\omega) = \#P^{(n)}(\omega) \le \#P_{\mathcal{N}(n')-1}(\omega) + 1.$$

Combined with Inequality 4.3.3, this yields the desired conclusion. □

We now arrive at the first description of the escape rate.

Corollary 4.3.2. *Let ω be a non-elementary random walk on G. Then there exists $K > 0$ such that*

$$\mathbb{P}\left(d(o, \omega_n\, o) \le Kn\right) \le \frac{1}{K} e^{-Kn}.$$

Proof. Lemma 4.1.1 tells us that there exists a sequence of Schottky axes $(\kappa_l)_{l=1}^M$ with $M > 4\#\,\mathcal{P}_n(\omega)$ such that $(o, \kappa_1, \ldots, \kappa_M, \omega_n\, o)$ is D_0-aligned. Proposition 3.1.5 then tells us that

$$d(o, \omega_n) \ge \left[\left(\frac{M_0}{K_0} - K_0\right) - 3E_0\right] \cdot (4\#\,\mathcal{P}_n(\omega)) \ge 4E_0 \#\,\mathcal{P}_n(\omega).$$

By combining this with Lemma 4.3.1, we arrive at the desired conclusion. □

Corollary 4.3.2 even implies

$$\mathbb{P}\left(\min\{\#\,\mathcal{P}_k(\omega) : k \ge n\} \le Kn\right) \le \frac{1}{K} e^{-Kn}$$

for some $K > 0$. We now claim that if $\# \mathcal{P}_k(\omega) \geq Kn$ for all $k \geq n$, then $\mathcal{P}_n(\omega), \mathcal{P}_{n+1}(\omega), \ldots$ all possess the same first $Kn-1$ pivotal times. Suppose to the contrary that for some $k \geq n$, $\mathcal{P}_k(\omega)$ does not start with the first $Kn-1$ pivotal times of $\mathcal{P}_n(\omega)$.

Since the complete set of pivotal times $P^{(n)}(\omega)$ has at least Kn elements, $P_{\mathcal{N}(n')-1}(\omega)$ has at least $Kn-1$ elements. Let $i_1, \ldots, i_{\lceil Kn-1 \rceil}$ be the first $\lceil Kn-1 \rceil$ elements of $P_{\mathcal{N}(n')-1}$.

We now note that one of $\{P_{\mathcal{N}(n')}(\omega), \ldots, P_{\mathcal{N}(k')-1}(\omega), P^{(k)}(\omega)\}$ becomes a proper subset of $\{i_1, \ldots, i_{\lceil Kn-1 \rceil}\}$; otherwise all of $i_1, \ldots, i_{\lceil Kn-1 \rceil}$ survives in $P^{(k)}(\omega)$ and leads to a contradiction. If $P^{(k)}$ is so, then we have a contradiction $\# \mathcal{P}_k(\omega) = \# P^{(k)}(\omega) < Kn-1$. Now suppose that $P_l(\omega)$ is so for some $\mathcal{N}(n') \leq l < k$. Since $P_l(\omega) = P^{(4M_0\, \vartheta(l))}(\omega)$, we have

$$\# \mathcal{P}_{4M_0\, \vartheta(l)}(\omega) = \# P_l < Kn-1, \tag{4.3.4}$$

$$\# \mathcal{P}_{4M_0\, \vartheta(l+1)}(\omega) = \# P_{l+1} \leq P_l(\omega) + 1 < Kn. \tag{4.3.5}$$

Note that $4M_0\, \vartheta(l+1) > n$, since otherwise we have a contradiction, namely, $\mathcal{N}(\lfloor n/4M_0 \rfloor) \geq l+1 > \mathscr{B}(n')$. However, Inequality 4.3.5 then also contradicts the assumption. Hence, the claim follows.

Having the argument above in mind, we define

$$\mathcal{Q}_n(\omega) := \cap_{k \geq n} \mathcal{P}_k(\omega), \quad \mathcal{Q}(\omega) := \cup_n \mathcal{Q}_n(\omega) = \liminf \mathcal{P}_n(\omega)$$

; we call this the *set of eventual pivotal times.* We have proven:

Lemma 4.3.3.

$$\mathbb{P}(\# \mathcal{Q}_n(\omega) \leq Kn) \leq \frac{1}{K} e^{-Kn} \tag{4.3.6}$$

holds for some $K > 0$.

Suppose now that $\# \mathcal{Q}_n(\omega) = \{i(1) < \ldots < i(M)\}$. Let $(\kappa_l)_{l=1}^{4M}$ be the sequence of Schottky axes at pivotal times in $\mathcal{Q}_n(\omega)$. Then for any $k, k' \geq n$,

$$(o, \kappa_1, \ldots, \kappa_{4M}, \omega_k\, o), \ (o, \kappa_1, \ldots, \kappa_{4M}, \omega_{k'}\, o)$$

are subsequences of D_0-aligned sequences; namely, they are D_1-aligned. Then the terminating point $\omega_{i(M)+4M_0}\, o$ of the last axes κ_{4M} is far from o and passed by $[o, \omega_k\, o]$ and $[o, \omega_{k'}\, o]$. More precisely, we have

$$d(o, \omega_{i(M)+4M_0}\, o), d(\omega_{i(M)+4M_0}\, o, o) \geq \left[\left(\frac{M_0}{K_0} - K_0 \right) - 3E_0 \right] \cdot 4M \geq 4E_0 M$$

and

$$d(\omega_{i(M)+4M_0}\, o, [\omega_k\, o, o]), d(\omega_{i(M)+4M_0}\, o, [o, \omega_{k'}\, o]) \leq E_0.$$

This implies that the Gromov product $(\omega_k\, o, \omega_{k'}\, o)_o$ is at least $4E_0 M - 4E_0$, and we have:

Corollary 4.3.4 ([Gou21, Proposition 4.13])**.** *There exists $K > 0$ such that the following hold:*

$$\mathbb{P}\left(\inf_{k,k' \geq n} (\omega_k\, o, \omega_{k'}\, o)_o \leq Kn \right) \leq \frac{1}{K} e^{-n/K}. \tag{4.3.7}$$

4.4 Some other models

Later, we will use two other models. The first model uses the same framework but with a different decomposition. Namely, first fix another probability measure μ' such that $0 \leq \mu' \leq c\mu^{M'}$ holds for some $c, M' > 0$, and consider:

$$\mu^{(4M_0+M')} = \alpha(\mu_S^2 \times \mu' \times \mu_S^2) + (1-\alpha)\nu$$

for some $0 < \alpha < 1$ and ν. ρ_i, η_i, ν_i are defined analogously.

The second model begins with the decomposition

$$\mu^{2M_0} = \alpha(\mu_S \times \mu_S) + (1-\alpha)\nu \tag{4.4.1}$$

for some $0 < \alpha < 1$ and *non-elementary* ν. This time we consider:

- Bernoulli RVs ρ_i with $\mathbb{P}(\rho_i = 1) = \alpha$ and $\mathbb{P}(\rho_i = 0) = 1 - \alpha$,
- η_i with the law of μ_S^2,
- ν_i with the law of ν, and
- ξ_i with the law of μ^{2M_0},

all independent. Fixing a large constant $K_{sleep} > 0$, we define a family $\{t_j, t_j'\}_{j=1}^{\infty}$ of RVs as follows. t_1 is the first time i with $\rho_i = 1$, and $t_1' := \min\{i > t_1 + K_{sleep} : \rho_i = 1\}$. Inductively, we define

$$t_k := \min\{i > t_{k-1}' : \rho_i = 1\}, \quad t_k' := \min\{i > t_k + K_{sleep} : \rho_i = 1\}.$$

We then define

$$(g_{2M_0k+1}, \ldots, g_{2M_0k+2M_0}) := \begin{cases} \eta_k & \text{when } k \in \{t_j, t_j'\}_{j=1}^{\infty} \\ \xi_k & \text{when } t_j + 1 \le k \le t_j + K_{sleep} \text{ for some } j \\ \nu_k & \text{otherwise.} \end{cases}$$

Then $(g_i)_{i=1}^{\infty}$ has the law of μ^{∞} [Gou21, Claim 5.11]. We also set $\omega_k := g_1 \cdots g_k$. This time, however, we define

$$\mathcal{N}(k) := \#\{j \ge 1 : t_j' < k\}.$$

Chapter 5. Deviation inequalities

In this chapter, we establish deviation inequalities. In order to derive deviation inequalities, we seek an (eventual) pivotal time at which the Schottky segment will witness two sides of the triangle made by points. This will make the triangle 'thin' and guarantee that the Gromov product is bounded by the progress made till the pivotal time. Such a pivotal time will appear before the n-th step outside a set of exponentially decaying probability. Using this exponential bound, we will estimate the p-moment and the $2p$-moment of the Gromov product.

5.1 Pivoting for a pair of independent paths

In this section, together with the K_0-Schottky set S, we consider its reflection

$$\check{S} := \{s^{-1} : s \in S\} = \{(\phi_{M_0}^{-1}, \ldots, \phi_1^{-1}) : (\phi_1, \ldots, \phi_{M_0}) \in S\}.$$

As in Section 4.1, we fix isometries $(w_j)_{j=0}^{\infty}$, $(v_j)_{j=1}^{\infty}$, $(\check{w}_j)_{j=0}^{\infty}$ and $(\check{v}_j)_{j=1}^{\infty}$. We then draw choices $s = (\alpha_j, \beta_j, \gamma_j, \delta_j)_{j=1}^{n} \in S^{4n}$ and $\check{s} = (\check{\alpha}_j, \check{\beta}_j, \check{\gamma}_j, \check{\delta}_j)_{j=1}^{n} \in \check{S}^{4n}$, and construct the set of pivotal times on the words

$$\begin{aligned} w &= w_0 a_1 b_1 v_1 c_1 d_1 \cdots a_n b_n v_n c_n d_n w_n, \\ \check{w} &= \check{w}_0 \check{a}_1 \check{b}_1 \check{v}_1 \check{c}_1 \check{d}_1 \cdots \check{a}_n \check{b}_n \check{v}_n \check{c}_n \check{d}_n \check{w}_n \end{aligned}$$

separately. Here, recall the notations $w_{i,j}^{\pm}$ and $\Upsilon(\alpha_j), \ldots, \Upsilon(\delta_j)$; analogous notations are employed for the path $\check{\omega}$, e.g., $\Upsilon(\check{\alpha}_1) = \check{w}_0 \Gamma(\alpha_1)$, $\Upsilon(\check{\beta}_1) = \check{w}_0 \check{a}_1 \Gamma(\beta_1)$, etc.

Let $\mathcal{E}$, $\check{\mathcal{E}}$ be equivalence classes made by the pivoting for ω and $\check{\omega}$, respectively. Let also

$$P(\mathcal{E}) = \{i(1) < i(2) < \ldots\}, \quad P(\check{\mathcal{E}}) = \{\check{i}(1) < \check{i}(2) < \ldots\}.$$

We will now construct

$$\begin{aligned} S_1^*(\check{s}, s) &:= S_1^*, \\ \check{S}_1^*(\check{s}, s) &:= \check{S}_1^*(\alpha_{\check{i}(1)}), \\ S_2^*(\check{s}, s) &:= \check{S}_2^*(\check{\alpha}_{\check{i}(1)}, \check{\beta}_{\check{i}(1)}, \check{\gamma}_{\check{i}(1)}, \alpha_{i(1)}, \beta_{i(1)}, \gamma_{i(1)}), \\ \check{S}_2^*(\check{s}, s) &:= \check{S}_2^*(\check{\alpha}_{\check{i}(1)}, \check{\beta}_{\check{i}(1)}, \check{\gamma}_{\check{i}(1)}, \alpha_{i(1)}, \beta_{i(1)}, \gamma_{i(1)}, \alpha_{i(2)}), \\ &\vdots \end{aligned}$$

for $1 \le k \le M$. We first consider

$$\phi_k := (\check{w}_{\check{i}(k),2}^{-})^{-1} w_{i(k),2}^{-} = \check{w}_{\check{i}(k)}^{-1} \check{d}_{\check{i}(k)-1}^{-1} \check{c}_{\check{i}(k)-1}^{-1} \cdots \check{w}_0^{-1} \cdot w_0 a_1 b_1 v_1 c_1 d_1 \ldots w_{i(k)}.$$

Then $S_k^*(\check{s}, s)$ and $\check{S}_k^*(\check{s}, s)$ are defined as follows.

$$\begin{aligned} S_k^*(\check{s}, s) &:= \left\{\alpha_{i(k)} \in S : \left(\phi_k^{-1} o, \Gamma(\alpha_{i(k)})\right) \text{ is } K_0\text{-aligned}\right\} \\ &:= \left\{\alpha_{i(k)} \in S : \left(\check{y}_{\check{i}(k),2}^{-}, \Upsilon(\alpha_{i(k)})\right) \text{ is } K_0\text{-aligned}\right\}, \\ \check{S}_k^*(\check{s}, s) &:= \left\{\check{\alpha}_{\check{i}(k)} \in S : \left(\phi_k a_{i(k)} o, \Gamma(\check{\alpha}_{\check{i}(k)})\right) \text{ is } K_0\text{-aligned}\right\} \\ &:= \left\{\check{\alpha}_{\check{i}(k)} \in S : \left(y_{i(k),1}^{-}, \Upsilon(\check{\alpha}_{\check{i}(k)})\right) \text{ is } K_0\text{-aligned}\right\}. \end{aligned}$$

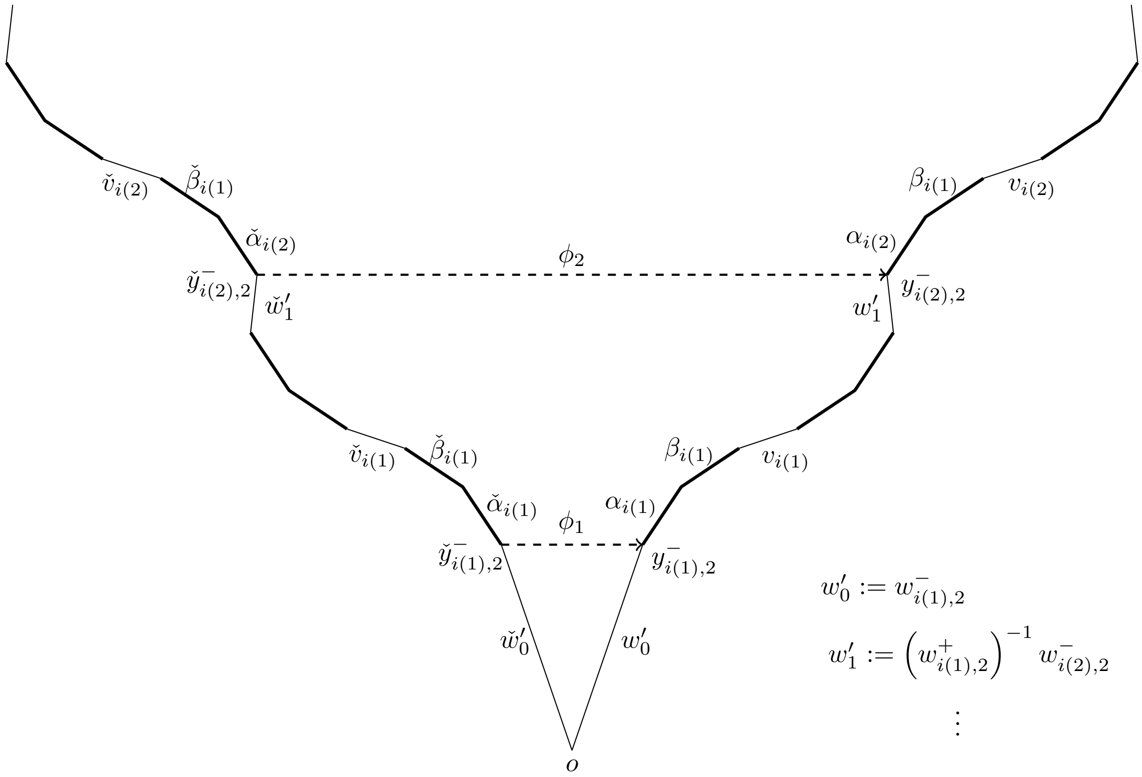

Figure 5.1: Defining ϕ_k's used in the pivoting for a pair of independent paths.

Then the property of Schottky sets imply that $S \setminus S^*_k$, $S \setminus \check{S}^*_k$'s consist of at most 1 element each. Moreover, Lemma 3.1.2 says that $\left(\bar{\Upsilon}(\check{\alpha}_{\check{i}(k)}), \Upsilon(\alpha_{i(k)})\right)$ is D_0-aligned when $\alpha_{i(k)} \in S^*_k$ and $\check{\alpha}_{\check{i}(k)} \in \check{S}^*_k$. Note also that $S^*_k(\check{s}, s)$, $\check{S}^*_k(\check{s}, s)$ depend only on the pivotal choices at the first $k-1$ pivotal times and independent from the pivoting later.

We now estimate the probability that $\alpha_{i(k)} \in S^*_k$ and $\check{\alpha}_{\check{i}(k)} \in \check{S}^*_k$. Given $s = (\alpha_{i(l)}, \beta_{i(l)}, \gamma_{i(l)})_{l=1,\ldots,k-1}$ and $\check{s} = (\check{\alpha}_{\check{i}(l)}, \check{\beta}_{\check{i}(l)}, \check{\gamma}_{\check{i}(l)})_{l=1,\ldots,k-1}$, we define

$$S^\dagger_k := \left\{ \begin{array}{c} (\alpha_{i(k)}, \beta_{i(k)}, \gamma_{i(k)}, \check{\alpha}_{\check{i}(k)}, \check{\beta}_{\check{i}(k)}, \check{\gamma}_{\check{i}(k)}) \in S_{i(k)}(\mathcal{E}) \times \check{S}_{\check{i}(k)}(\check{\mathcal{E}}) \\ : \ \alpha_{i(k)} \in S^*_k(\check{s}, s) \text{ and } \check{\alpha}_{\check{i}(k)} \in \check{S}^*_k(\check{s}, s, \check{\alpha}_{\check{i}(k)}) \end{array} \right\}$$

Then we have the following:

Lemma 5.1.1. *For each $1 \le k \le \lfloor M/2 \rfloor$, $S^\dagger_k$ has cardinality at least $(\#S)^6 - 8(\#S)^5$.*

Proof. There are at least $(\#S-1)$ choices of $\gamma_{i(k)}$ and $\check{\gamma}_{\check{i}(k)}$ that satisfy Inequality 4.1.2. Fixing those choices, at least $(\#S-1)$ choices of $\beta_{i(k)}$ and $\check{\beta}_{i(\check{k})}$ in S satisfy Inequality 4.1.4. Fixing those choices, there are at most 1 choice of $\alpha_{i(k)}$ in S that violates Inequality 4.1.5 and at most 1 choice that lies outside S^*_k. If we choose $\alpha_{i(k)}$ in S^*_k that satisfies Inequality 4.1.5, now at least $(\#S-2)$ choices of $\check{\alpha}_{\check{i}(k)}$ satisfy Inequality 4.1.5 and belong to $\check{S}^*_k$. Overall, we conclude that $S^*_{(k)}$ has cardinality at least $(\#S-1)^4(\#S-2)^2 \ge (\#S)^6 - 8(\#S)^5$. □

Corollary 5.1.2. *If $\#P_n(\mathcal{E})$, $\#P_n(\check{\mathcal{E}})$ are greater than m, then we have*

$$\mathbb{P}\left(\alpha_{i(k)} \in S^*_k(\check{s}, s), \check{\alpha}_{\check{i}(k)} \in \check{S}^*_k(\check{s}, s) \text{ for some } k \le m \,\middle|\, \mathcal{E} \times \check{\mathcal{E}}\right) \ge 1 - \left(\frac{8}{N_0}\right)^m. \tag{5.1.1}$$

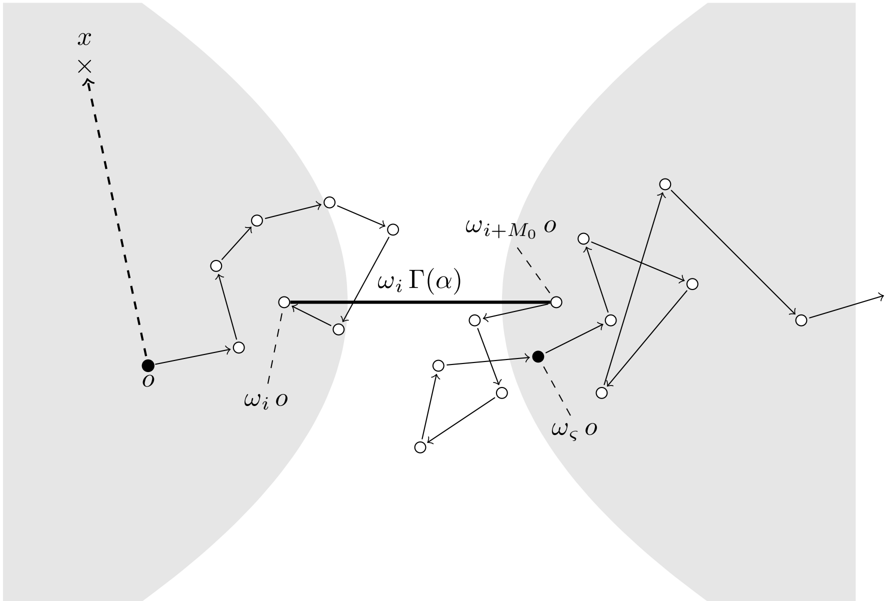

Figure 5.2: Persistent progress and ς. Here, o and x are on the left with respect to the persistent progress $\omega_i\,\Gamma(\alpha)$, while the loci after $\omega_\varsigma\, o$ are all on the right. Note that we do not restrict the locations of $\omega_1\, o, \ldots, \omega_{i-1}\, o$ and $\omega_{i+M_0+1}\, o, \ldots, \omega_{\varsigma-1}\, o$.

5.2 Persistent progress

Given $x \in X$, we seek an index k such that there exists $i \leq k - M_0$ such that:

1. $\alpha := (g_{i+1}, \ldots, g_{i+M_0})$ is a Schottky sequence;
2. $(o, \omega_i\, \Gamma(\alpha), \omega_n\, o)$ is D_1-aligned for all $n \geq k$;
3. $(x, \omega_i\, \Gamma(\alpha))$ is D_1-aligned.

Let $\varsigma = \varsigma(\omega; x)$ be the minimal index k that satisfies the above. If such an index does not exist, then we set $\varsigma = +\infty$.

For example, when $x = o$, $\varsigma(\omega; o)$ will be smaller than or equal to n if $\mathcal{Q}_n(\omega) \neq \emptyset$. We have previously constructed the pivotal times in order to guarantee witnessing of $[o, \omega_n\, o]$. We will now perform additional pivoting at the pivotal times to guarantee the witnessing of $[x, \omega_n\, o]$ as well.

Lemma 5.2.1. *There exists $K, \kappa > 0$ such that for any $x \in X$ and $g_{k+1} \in G$, we have*

$$\mathbb{P}\left(\varsigma(\omega; x) \geq k \,\middle|\, g_{k+1}\right) \leq K e^{-\kappa k}$$

for each k.

Proof. We first freeze the choices of $g_{4M_0\lfloor k/4M_0 \rfloor + 1}, \ldots, g_{4M_0(\lfloor k/4M_0 \rfloor + 1)}$ (or equivalently, the values of $\rho_{\lfloor k/4M_0 \rfloor}$, $\nu_{\lfloor k/4M_0 \rfloor}$ and $\eta_{\lfloor k/4M_0 \rfloor}$) and exclude them from the potential pivotal time. We still have $\mathbb{P}(\#\, \mathcal{Q}_k \leq \kappa_1 k) \leq K_1 e^{-\kappa_1 k}$.

Let us fix an equivalence class $\mathcal{E}$ made by pivoting the choice of β_i's at the first $\kappa_1 k$ eventual pivotal times that appeared before k. Let $i(1) < \ldots < i(\kappa_1 k)$ be the first $\kappa_1 k$ eventual pivotal times in $\mathcal{Q}_k(\mathcal{E})$, and $j(1) < \ldots < j(\kappa_1 k)$ be be the corresponding indices in the fixed words model, i.e., $4M_0\, \vartheta(j(l)) = i(l)$ for $l = 1, \ldots, \kappa_1 k$.

Recall that $\omega \in \mathcal{E}$ is then determined by the choices $(\beta_{j(1)}, \ldots, \beta_{j(\kappa_1 k)})$, and at each l there are at least $N_0 - 1$ choices of $\beta_{j(l)}$ for the pivoting. For any $\omega \in \mathcal{E}$ and $l = 1, \ldots, \kappa_1 k$, we have:

- $i(l) + 3M_0 \le k - M_0$,
- $\beta_{j(l)} = (g_{i(l)+M_0+1}, \ldots, g_{i(l)+2M_0})$ and $\gamma_{j(l)} = (g_{i(l)+2M_0+1}, \ldots, g_{i(l)+3M_0})$ are Schottky, and
- $(o, \Upsilon(\beta_{j(l)}), \omega_n o)$, $(o, \Upsilon(\gamma_{j(l)}), \omega_n o)$ are D_1-aligned for all $n \ge k$ by Lemma 4.1.1 and Proposition 3.1.4.

It now suffices to guarantee for most $\omega \in \mathcal{E}$ that $(x, \Upsilon(\beta_{j(l)}))$ or $(x, \Upsilon(\gamma_{j(l)}))$ is D_1-aligned at some l.

Suppose that $(x, \Upsilon(\gamma_{j(\kappa_1 k)}))$ is not D_1-aligned for some $\omega \in \mathcal{E}$ $(*)$. Recall:

$$\big(\Upsilon(\alpha_{j(1)}), \Upsilon(\beta_{j(1)}), \Upsilon(\gamma_{j(1)}), \ldots, \Upsilon(\gamma_{j(\kappa_1 k)})\big)$$

is a subsequence of a D_0-aligned Schottky axes. $(*)$ and Proposition 3.1.4 imply that $(\Upsilon(\beta_{j(l)}), x)$ is D_1-aligned for $l = 1, \ldots, \kappa_1 k$. In particular, $(x, \Upsilon(\beta_{j(l)}))$ is not D_1-aligned for $l = 1, \ldots, \kappa_1 k$. Let us now consider

$$\tilde{\omega} = (\tilde{\beta}_{j(1)}, \ldots, \tilde{\beta}_{j(1)}) \in \mathcal{E}$$

that differs from ω. Let $j(l)$ be the first index at which ω and $\tilde{\omega}$ differ. Then $\omega_{i(l)+M_0} = \tilde{\omega}_{i(l)+M_0}$ holds, and $(x, \Upsilon(\tilde{\beta}_{j(l)}))$ is K_0-aligned by the property of the Schottky set S. Therefore, we have either:

- $(x, \Upsilon(\gamma_{j(\kappa_1 k)}))$ is D_1-aligned for all $\omega \in \mathcal{E}$, or;
- $(x, \Upsilon(\beta_{j(l)}))$ is K_0-aligned at some l for all but one $\omega \in \mathcal{E}$.

In summary, we have

$$\mathbb{P}\left(\varsigma(\omega; x) \ge k \,\middle|\, \mathcal{E}\right) \le \frac{1}{\#\mathcal{E}} \le \left(\frac{1}{N_0 - 1}\right)^{\kappa_1 k} \le \left(\frac{2}{N_0}\right)^{\kappa_1 k}.$$

These conditional probabilities and the probability $\mathbb{P}\{\#\, \mathcal{Q}_k(\omega) \le \kappa_1 k\}$ together take up an exponentially decaying probability. $\square$

For $\omega \in \Omega$ and $n, k \ge \varsigma(\omega; x)$, we have i such that

1. $\alpha := (g_{i+1}, \ldots, g_{i+M_0})$ is a Schottky sequence;
2. $(o, \omega_i \,\Gamma(\alpha), \omega_n o)$ and $(o, \omega_i \,\Gamma(\alpha), \omega_k o)$ are D_2-aligned, and
3. $(x, \omega_i \,\Gamma(\alpha))$ is D_2-aligned.

By Proposition 3.1.5, there exists $q \in [x, \omega_n o]$ that are within d-distance E_0 from $\omega_i o$. Hence, we have

$$\begin{aligned}(x, \omega_n o)_o &\le \frac{1}{2}\left[\begin{array}{c} d(x, \omega_i o) + d(\omega_i o, o) + d(o, \omega_i o) + d(\omega_i o, \omega_n o) \\ -d(x, q) - d(q, \omega_n o) \end{array}\right] \\ &\le d(o, \omega_i o) + d(q, \omega_i o) < d(o, \omega_k o).\end{aligned}$$

Here, the final inequality holds because $[o, \omega_k o]$ is E_0-witnessed by $[\omega_i o, \omega_{i+M_0} o]$ whose length is at least $10E_0$.

For a similar reason, we have $d(o, [x, \omega_n o]) \le d(o, \omega_\varsigma o)$. Hence, we obtain:

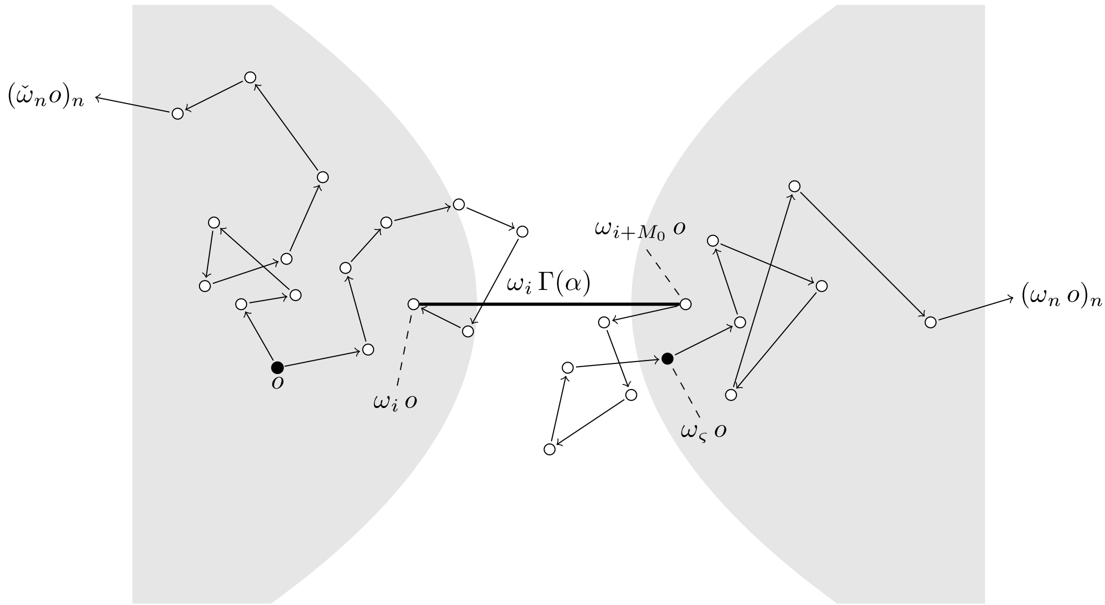

Figure 5.3: Persistent progress and υ. Here, all of the backward loci $(\check{\omega}_n o)_{n\geq 0}$ are on the left of the persistent progress $\omega_i\, \Gamma(\alpha)$, while the forward loci after $\omega_\varsigma\, o$ are all on the right.

Corollary 5.2.2. *There exist $\kappa, K > 0$ such that for any $x \in X$ and $g_{k+1} \in G$, we have*

$$\mathbb{P}\left[\sup_{n\geq k}(x, \omega_n\, o)_o \geq d(o, \omega_k\, o) \,\middle|\, g_{k+1}\right] \leq K e^{-\kappa k},$$
$$\mathbb{P}\left[\sup_{n\geq k} d(o, [x, \omega_n\, o]) \geq d(o, \omega_k\, o) \,\middle|\, g_{k+1}\right] \leq K e^{-\kappa k}.$$

Let us now define another index for a persistent progress made by two independent paths $(\check{\omega}, \omega)$. Given k, we seek an index $i \leq k - M_0$ such that:

1. $\alpha := (g_{i+1}, \ldots, g_{i+M_0})$ is a Schottky sequence;
2. $(o, \omega_i\, \Gamma(\alpha), \omega_n\, o)$ is D_1-aligned for all $n \geq k$, and
3. $(\check{\omega}_{n'}\, o, \omega_i\, \Gamma(\alpha))$ is D_2-aligned for all $n' \geq 0$.

We define $\upsilon = \upsilon(\check{\omega}, \omega)$ by the minimal index k such that the above index $i \leq k$ exists. In other words, after index k, the forward path ω deviates forever from the directions made by each point in the backward path $\check{\omega}$. Moreover, this deviation is witnessed by some Schottky progress $\omega_i\, \Gamma(\alpha)$ made before index k.

Lemma 5.2.3. *There exist $\kappa, K > 0$ such that the following estimate holds for all k:*

$$\mathbb{P}\left(\upsilon(\check{\omega}, \omega) \geq k \,\middle|\, g_{k+1}, \check{g}_1, \ldots, \check{g}_{k+1}\right) \leq K e^{-\kappa k}. \tag{5.2.1}$$

Proof. We first freeze the choices of $g_{4M_0\lfloor k/4M_0\rfloor+1}, \ldots, g_{4M_0(\lfloor k/4M_0\rfloor+1)}$ and $\check{g}_1, \ldots, \check{g}_{4M_0\lceil (k+1)/4M_0\rceil}$. We still have $\mathbb{P}(\#\, \mathcal{Q}_k(\omega) \leq \kappa_1 k) \leq K_1 e^{-\kappa_1 k}$ and $\mathbb{P}(\#\, \mathcal{Q}_{2k}(\check{\omega}) \leq \kappa_1 k) \leq K_1 e^{-\kappa_1 k}$.

Now for paths ω with $\mathcal{Q}_k(\omega) > \kappa_1 k$, we pivot at the first $\kappa_1 k$ pivotal times; let $\mathcal{E}$ be one equivalence class made from this early pivoting. Let also $\check{\mathcal{E}}$ be an equivalence class of backward paths $\check{\omega}$'s that have $\#\, \mathcal{Q}_{2k}(\check{\omega}) \geq \kappa_1 k$, made by pivoting at the first $\kappa_1 k$ pivotal times. Note that the pivotal times for $\check{\omega}$'s are always formed after $k+1$ since we have frozen the first $4M_0\lceil (k+1)/4M_0 \rceil$ steps. Let

$$\begin{aligned} \mathcal{Q}_k(\mathcal{E}) &= \{i(1) < \ldots < i(\kappa_1 k) < \ldots\}, \\ \mathcal{Q}_{2k}(\check{\mathcal{E}}) &= \{\check{i}(1) < \ldots < \check{i}(\kappa_1 k) < \ldots\}, \\ i(l) = 4M_0\,\vartheta(j(l)), &\quad \check{i}(l) = 4M_0\check{\vartheta}(\check{j}(l)) \quad (l = 1, \ldots, \kappa_1 k). \end{aligned}$$

Now on $\check{\mathcal{E}} \times \mathcal{E}$, Corollary 5.1.2 implies that $(\bar{\Upsilon}(\check{\alpha}_{\check{j}(l)}), \Upsilon(\alpha_{j(l)}))$ is K_0-aligned for some $l \le \kappa_1 k/2$ for probability at least $1 - (8/N_0)^{\kappa_1 k/2}$ on $\check{\mathcal{E}} \times \mathcal{E}$. We now freeze the choices at the first $\kappa_1 k/2$ pivotal times for ω and the entire pivotal times for $\check{\omega}$ that make $(\bar{\Upsilon}(\check{\alpha}_{\check{j}(l)}), \Upsilon(\alpha_{j(l)}))$ K_0-aligned. Then $\mathcal{E}$ is divided into finer equivalence classes $\mathcal{E}_1$ made by pivoting at the latter $\kappa_1 k/2$ pivotal times for ω.

Lemma 5.2.1 asserts that for each $n' = 1, 2, \ldots, 2k$, $(\check{\omega}_{n'}\, o, \Upsilon(\gamma_{j(\kappa_1 k)}))$ is D_1-aligned for all but at most one choice in $\mathcal{E}_1$. Except at most $2k$ such bad choices, we now have the following:

- $i(\kappa_1 k) + 4M_0 \le k$,
- $(o, \Upsilon(\gamma_{j(\kappa_1 k)}), \omega_n\, o)$ is D_2-aligned for all $n \ge k$,
- $\big(\check{\omega}_{n'}\, o, \bar{\Upsilon}(\check{\gamma}_{j(l)}), \Upsilon(\alpha_{j(l)}), \Upsilon(\gamma_{j(\kappa_1 k)})\big)$ is a subsequence of a D_1-aligned sequence for all $n' \ge 2k$, and
- $\big(\check{\omega}_{n'}\, o, \Upsilon(\gamma_{j(\kappa_1 k)})\big)$ is D_2-aligned for $n' = 1, \ldots, 2k$.

Then $(\check{\omega}_{n'} o, \Upsilon(\gamma_{j(\kappa_1 k)}))$ is D_2-aligned for all n' by Proposition 3.1.4, and $i(\kappa_1 k) + 2M_0 \le k - M_0$ works for ω. Hence,

$$\mathbb{P}\left(\upsilon(\check{\omega}, \omega) \ge k \,\middle|\, \check{\mathcal{E}} \times \mathcal{E}\right) \le \left(\frac{8}{N_0}\right)^{\kappa_1 k/2} + 2k \cdot \left(\frac{3}{N_0}\right)^{\kappa_1 k/2}.$$

We now sum up these conditional probabilities and the excluded probability to conclude. □

As before, we deduce

$$(\check{\omega}_{n'} o, \omega_n\, o)_o \le d(o, \omega_k\, o)$$

for all $n' \ge 0$ and $n, k \ge \upsilon(\check{\omega}, \omega)$. Hence, we deduce:

Corollary 5.2.4. *There exist $\kappa, K > 0$ such that for any $g_{k+1}, \check{g}_1, \ldots, \check{g}_{k+1} \in G$, we have*

$$\mathbb{P}\left[\sup_{n' \ge 0, n \ge k} (\check{\omega}_{n'} o, \omega_n\, o)_o \ge d(o, \omega_k\, o) \,\middle|\, g_{k+1}, \check{g}_1, \ldots, \check{g}_{k+1}\right] \le K e^{-\kappa k}.$$

We similarly define $\check{\upsilon} = \check{\upsilon}(\check{\omega}, \omega)$ as the minimal index k that are associated with another index $i \le k$ such that:

1. $\check{\alpha} := (\check{g}_{i+1}, \ldots, \check{g}_{i+M_0})$ is a Schottky sequence;
2. $(o, \check{\omega}_i \Gamma(\check{\alpha}), \check{\omega}_n o)$ is D_1-aligned for all $n \ge k$, and
3. $(\omega_n\, o, \check{\omega}_i \Gamma(\check{\alpha}))$ is D_2-aligned for all $n \ge 0$.

Then we similarly have

$$\mathbb{P}\left(\check{\upsilon}(\check{\omega}, \omega) \ge k \,\middle|\, \check{g}_{k+1}, g_1, \ldots, g_{k+1}\right) \le K_2 e^{-\kappa_2 k}. \tag{5.2.2}$$

Note that Inequality 5.2.1 is proven using the pivoting at the first k steps of ω and eventual escape to infinity of ω, $\check{\omega}$. This enables us to fix $\check{g}_1, \ldots, \check{g}_{k+1}$ and g_{k+1} in prior: we do not use the randomness of the initial trajectory of $\check{\omega}$. Likewise, Inequality 5.2.2 does not rely on the pivoting at the initial k steps of ω. This will lead to the exponent doubling for the geodesic tracking; roughly speaking, this is a consequence of the fact that the minimum of two independent RVs with finite p-th moment has finite $2p$-th moment.

5.3 Deviation inequalities

Thanks to Corollary 5.2.2 and 5.2.4, we can establish the following deviation inequality.

Proposition 5.3.1. *Suppose that μ has finite p-moments for some $p > 0$. Then there exists $K > 0$ such that for any $x \in X$, we have*

$$\mathbb{E}\left[\sup_{n\geq 0}(x, \omega_n\, o)_o^p\right], \quad \mathbb{E}\left[\sup_{n,n'\geq 0}(\check{\omega}_{n'} o, \omega_n\, o)_o^{2p}\right] < K.$$

Proof. We have observed that $\sup_{n\geq\varsigma(\omega;x)}(x, \omega_n\, o)_o$ is dominated by $d(o, \omega_{\varsigma(\omega;x)}\, o)$. Moreover, for $i = 1, \ldots, \varsigma(\omega; x)$, $(x, \omega_i\, o)_o$ and $(\omega_i\, o, x)_o$ are bounded above by $d(o, \omega_i\, o)$. Hence, we have

$$\begin{aligned}\sup_n (x, \omega_n\, o)_o^p &\leq \max_{1\leq i\leq\varsigma(\omega;x)} d(o, \omega_i\, o)^p \\ &\leq \sum_{i=0}^{\infty} |d(o, \omega_{i+1}\, o)^p - d(o, \omega_i\, o)^p|\, 1_{i<\varsigma(\omega;x)}.\end{aligned}$$

Let us now recall that two simple inequalities: for $t, s \geq 0$,

$$|t^p - s^p| \leq \begin{cases} |t-s|^p & p \leq 1, \\ 2^p\left(|t-s|^p + s^{p-1}|t-s|\right) & p > 1. \end{cases} \tag{5.3.1}$$

Moreover, for $t_1, \ldots, t_n \geq 0$ and $p > 0$, we have

$$(t_1 + \ldots + t_n)^p \leq (n \max_i t_i)^p \leq n^p(t_1^p + \ldots + t_n^p)$$

and

$$\mathbb{E}[d(o, \omega_n\, o)^p] \leq n^{p+1}\, \mathbb{E}_\mu[d(o, go)^p].$$

Hence, it suffices to show that

$$\mathbb{E}\left[\sum_{i=0}^{\infty} d(o, g_{i+1}o)^p 1_{i<\varsigma(\omega;x)}\right] < K_1$$

for some K_1 that does not depend on x, and when $p > 1$, also

$$\mathbb{E}\left[\sum_{i=0}^{\infty} d(o, \omega_i\, o)^{p-1} d(o, g_{i+1}o) 1_{i<\varsigma(\omega;x)}\right] < K_2$$

for some K_2 that does not depend on x.

The first summation is estimated based on Lemma 5.2.1. Let K_3, κ_3 be as in Lemma 5.2.1; recall that K_3, κ_3 does not depend on x. Then we have

$$\begin{aligned}\sum_{i=0}^{\infty} \mathbb{E}\left[d(o, g_{i+1}o)^p 1_{i<\varsigma}\right] &= \sum_{i=0}^{\infty} \mathbb{E}\left[d(o, g_{i+1}o)^p \cdot \mathbb{P}\left(\varsigma(\omega; x) > i \,\middle|\, g_{i+1}\right)\right] \\ &\leq \sum_{i=0}^{\infty} \mathbb{E}\left[d(o, g_{i+1}o)^p \cdot K_3 e^{-\kappa_3 i}\right] \\ &\leq 2^p \left(\mathbb{E}_\mu[d(o, go)^p] + \mathbb{E}_{\check{\mu}}[d(o, go)^p]\right) \cdot K_3 \sum_i e^{-\kappa_3 i} =: K_1\end{aligned}$$

Similarly, for $p > 1$, we estimate based on a dichotomy. Note that for any g_{i+1} and $c > 0$, we have

$$\begin{aligned}\mathbb{E}\left[d(o, \omega_i\, o)^{p-1} 1_{\varsigma>i} \,\middle|\, g_{i+1}\right] &\leq \mathbb{E}\left[d(o, \omega_i\, o)^{p-1} 1_{\varsigma>i} 1_{d(o,\omega_i\, o)\leq c} \,\middle|\, g_{i+1}\right] \\ &\quad + \mathbb{E}\left[d(o, \omega_i\, o)^{p-1} 1_{\varsigma>i} 1_{d(o,\omega_i\, o)>c} \,\middle|\, g_{i+1}\right] \\ &\leq c^{p-1}\, \mathbb{P}\left(\varsigma > i \,\middle|\, g_{i+1}\right) + \mathbb{E}\left[d(o, \omega_i\, o)^p \cdot c^{-1} | g_{i+1}\right] \\ &\leq c^{p-1} K_3 e^{-\kappa_3 i} + c^{-1} i^{p+1} \cdot \mathbb{E}_\mu[d(o, go)^p].\end{aligned} \tag{5.3.2}$$

By setting $c = e^{\kappa_3 i/2p}$, we deduce

$$\begin{aligned}
&\sum_{i=0}^{\infty} \mathbb{E}\left[d(o,\omega_i\, o)^{p-1} d(o, g_{i+1}o) 1_{i<\varsigma}\right] \\
&= \sum_{i=0}^{\infty} \mathbb{E}\left[d(o,g_{i+1}o)\, \mathbb{E}\left[d(o,\omega_i\, o)^p 1_{\varsigma>i} \,\middle|\, g_{i+1}\right]\right] \\
&\le \sum_{i=0}^{\infty} \mathbb{E}\left[d(o,g_{i+1}o)\cdot\left(K_3 e^{-\kappa_3 i/2} + i^{p+1} e^{-\kappa_3 i/2p}\, \mathbb{E}_\mu[d(o,go)^p]\right)\right] \\
&\le \left(K_3\, \mathbb{E}_\mu[d(o,go)^p] + \mathbb{E}_\mu[d(o,go)^p]^2\right)\cdot \sum_i i^{p+1} e^{-\kappa i/2p} =: K_2.
\end{aligned}$$

Clearly, K_1 and K_2 do not depend on the choice of x.

We now investigate $(\check{\omega}_{n'} o, \omega_n\, o)_o$. Let

$$\check{D}_k := \sum_{i=1}^{k} d(o, \check{g}_i o), \quad D_k := \sum_{i=1}^{k} d(o, g_i o).$$

It is clear that $d(o, \omega_k\, o) < D_l$ for all $k \le l$.

We begin by claiming that

$$\sup_{n',n\ge 0} (\check{\omega}_{n'} o, \omega_n\, o)_o^{2p} \le \sum_{i=0}^{\infty} |\check{D}_{i+1}^p D_{i+1}^p - \check{D}_i^p D_i^p| \left(1_{\check{D}_i \ge D_i} 1_{i<\upsilon} + 1_{\check{D}_i \le D_i} 1_{i<\check{\upsilon}}\right).$$

First, note that the RHS is at least $\check{D}_l^p D_l^p$ for

$$l := \min\left\{i : 1_{\check{D}_i \ge D_i} 1_{i<\upsilon} + 1_{\check{D}_i \le D_i} 1_{i<\check{\upsilon}} = 0\right\}.$$

(If such minimum does not exist, then the RHS becomes infinity almost surely since $\check{D}_k$, D_k tends to infinity almost surely.) Note that either $\check{D}_l \ge D_l$ or $\check{D}_l \le D_l$ holds.

In the first case $l \ge \upsilon$ must hold. Then for $n' \ge 0$ and $n \ge l$, we have

$$(\check{\omega}_{n'} o, \omega_n\, o)_o^{2p} \le d(o, \omega_l\, o)^{2p} \le D_l^{2p} \le \check{D}_l^p D_l^p.$$

Moreover, for $n' \ge 0$ and $n \le l$, we have

$$(\check{\omega}_{n'} o, \omega_n\, o)_o^{2p} \le d(o, \omega_n\, o)^{2p} \le D_n^{2p} \le D_l^{2p} \le \check{D}_l^p D_l^p.$$

In the second case $l \ge \check{\upsilon}$ must hold, and the argument as above implies that $(\check{\omega}_{n'} o, \omega_n\, o)_o^{2p}$ is dominated by $\check{D}_l^p D_l^p$, as desired.

Note that for $t_i, s_i \ge 0$, we have

$$\begin{aligned}
|t_1^p t_2^p - s_1^p s_2^p| &= |t_1^p (t_2^p - s_2^p) + (t_1^p - s_1^p) s_2^p| \\
&\le 2^{p+q}\left(|t_1 - s_1|^p + s_1^{p-n_p}|t_1 - s_1|^{n_p} + s_1^p\right)\left(|t_2 - s_2|^p + s_2^{p-n_p}|t_2 - s_2|^{n_p}\right) \\
&\quad + 2^p\left(|t_1 - s_1|^p + s_1^{p-n_p}|t_1 - s_1|^{n_p}\right) s_2^p. \\
&\left(n_p = \begin{cases} p & 0 \le p \le 1 \\ 1 & p > 1. \end{cases}\right)
\end{aligned}$$

Considering this, it suffices to show

$$\mathbb{E}\left[d(o, \check{g}_{i+1})^{n_1} d(o, g_{i+1})^{n_2} \check{D}_i^{p-n_1} D_i^{p-n_2} \left(1_{\check{D}_i \ge D_i} 1_{i<\upsilon} + 1_{\check{D}_i \le D_i} 1_{i<\check{\upsilon}}\right)\right] < K(i+1)^{2p+2} e^{-\kappa i}$$

for some K and κ, for $0 \le n_1, n_2 \le p$ such that $n_1 + n_2 \ge \min(p, 1)$. We will discuss the case $n_2 > 0$; the other case can be handled in the same way.

Let us first fix $\check{g}_{i+1}$ and g_{i+1}. We then compute

$$\begin{aligned}
&\mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{\check{D}_i \ge D_i} 1_{i<\upsilon} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] \\
&\le \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{D_i > c} 1_{\check{D}_i \ge D_i} 1_{i<\upsilon} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] + \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{D_i \le c} 1_{\check{D}_i \ge D_i} 1_{i<\upsilon} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] \\
&\le \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p} \cdot c^{-n_2} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] + \mathbb{E}\left[\check{D}_i^{p-n_1} \cdot \mathbb{E}\left[c^{p-n_2} 1_{i<\upsilon} \,\middle|\, \check{g}_1, \ldots, \check{g}_{i+1}, g_{i+1}\right]\right] \\
&\le \mathbb{E}[\check{D}_i^{p-n_1}] \cdot \mathbb{E}[D_i^p] \cdot c^{-n_2} + \mathbb{E}[\check{D}_i^{p-n_1}] \cdot c^{p-n_2}\, \mathbb{P}\left[\upsilon > i \,\middle|\, \check{g}_1, \ldots, \check{g}_{i+1}, g_{i+1}\right] \\
&\le (i+1)^{p-n_1+1}\, \mathbb{E}_\mu[d(o, go)^{p-n_1}] \cdot (i+1)^{p+1}\, \mathbb{E}_\mu[d(o, go)^p] \cdot c^{-n_2} \\
&\quad + (i+1)^{p-n_1+1}\, \mathbb{E}_\mu[d(o, go)^{p-n_1}] \cdot c^{p-n_2} \cdot K_3 e^{-\kappa_3 i}.
\end{aligned}$$

We also observe

$$\begin{aligned}
&\mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{\check{D}_i \le D_i} 1_{i<\check{\upsilon}} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] \\
&\le \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{\check{D}_i > c} 1_{\check{D}_i \le D_i} 1_{i<\check{\upsilon}} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] + \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{\check{D}_i \le c} 1_{\check{D}_i \le D_i} 1_{i<\check{\upsilon}} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] \\
&\le \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{D_i > c} 1_{i<\check{\upsilon}} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] + \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{\check{D}_i \le c} 1_{i<\check{\upsilon}} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] \\
&\le \mathbb{E}\left[\check{D}_i^{p-n_1} D_i^{p} \cdot c^{-n_2} \,\middle|\, \check{g}_{i+1}, g_{i+1}\right] + \mathbb{E}\left[D_i^{p-n_2} \cdot \mathbb{E}\left[c^{p-n_1} 1_{i<\check{\upsilon}} \,\middle|\, \check{g}_{i+1}, g_1, \ldots, g_{i+1}\right]\right] \\
&\le \mathbb{E}[\check{D}_i^{p-n_1}] \cdot \mathbb{E}[D_i^p] \cdot c^{-n_2} + \mathbb{E}[D_i^{p-n_2}] \cdot c^{p-n_1}\, \mathbb{P}\left[\check{\upsilon} > i \,\middle|\, \check{g}_{i+1}, g_1 \ldots, g_{i+1}\right] \\
&\le (i+1)^{p-n_1+1}\, \mathbb{E}_\mu[d(o, go)^{p-n_1}] \cdot (i+1)^{p+1}\, \mathbb{E}_\mu[d(o, go)^p] \cdot c^{-n_2} \\
&\quad + (i+1)^{p-n_2+1}\, \mathbb{E}_\mu[d(o, go)^{p-n_2}] \cdot c^{p-n_1} \cdot K_3 e^{-\kappa_3 i}.
\end{aligned}$$

Note that the trick

$$\check{D}_i^{p-n_1} D_i^{p-n_2} 1_{D_i > c} < \check{D}_i^{p-n_1} D_i^{p} c^{-n_2}$$

makes use of the fact $n_2 > 0$; it cannot work on the side of $\check{D}_i$ since n_1 may vanish in this case. Throughout the first argument, the factor $1_{\check{D}_i \ge D_i}$ did not play any role (though it is necessary for the case $n_1 > 0$ and $n_2 = 0$); the factor $1_{\check{D}_i \le D_i}$ in the second argument played a role only once, namely, switching $\check{D}_i$ and D_i at the second step.

The proof ends by taking $c = e^{\kappa_3 i/2p}$. $\square$

We now record a corollary for the geodesic tracking.

Corollary 5.3.2. *Suppose that μ has finite p-moment for some $p > 0$. Then there exists $K > 0$ such that*

$$\mathbb{E}\left[\min\{d(o, \omega_\upsilon\, o), d(o, \check{\omega}_{\check{\upsilon}} o)\}^{2p}\right] < K.$$

Proof. In view of the second half of the previous proof, it suffices to check

$$\min\{d(o, \omega_\upsilon\, o), d(o, \check{\omega}_{\check{\upsilon}} o)\}^{2p} \le \sum_{i=0}^{\infty} |\check{D}_{i+1}^p D_{i+1}^p - \check{D}_i^p D_i^p| \left(1_{\check{D}_i \ge D_i} 1_{i<\upsilon} + 1_{\check{D}_i \le D_i} 1_{i<\check{\upsilon}}\right).$$

The RHS is at least $\check{D}_l^p D_l^p$ for $l = \min\{i : 1_{\check{D}_i \ge D_i} 1_{i<\upsilon} + 1_{\check{D}_i \le D_i} 1_{i<\check{\upsilon}} = 0\}$. Note that either $\check{D}_l \ge D_l$ or $\check{D}_l \le D_l$ holds. In the first case, we are forced to have $l \ge \upsilon$; then

$$\min\{d(o, \omega_\upsilon\, o), d(o, \check{\omega}_{\check{\upsilon}} o)\}^{2p} \le d(o, \omega_\upsilon\, o)^{2p} \le D_\upsilon^{2p} \le D_l^{2p} \le \check{D}_l^p D_l^p.$$

In the second case, we are forced to have $l \ge \check{\upsilon}$; then

$$\min\{d(o, \omega_\upsilon\, o), d(o, \check{\omega}_{\check{\upsilon}} o)\}^{2p} \le d(o, \check{\omega}_{\check{\upsilon}} o)^{2p} \le \check{D}_{\check{\upsilon}}^{2p} \le \check{D}_l^{2p} \le \check{D}_l^p D_l^p$$

as desired. $\square$

We also discuss the case of finite exponential moment.

Proposition 5.3.3. *Suppose that μ has finite exponential moment. Then there exist $\kappa, K > 0$ such that*

$$\mathbb{E}\left[\sup_{n,n'\geq 0} e^{\kappa(x,\omega_n\, o)_o}\right] < K, \quad \mathbb{E}\left[\sup_{n,n'\geq 0} e^{\kappa(\check{\omega}_{n'}o,\omega_n\, o)_o}\right] < K.$$

Proof. We explain the latter inequality; the former one follows from the same argument by replacing the role of υ with ς.

Note that $(\check{\omega}'_n o, \omega_n\, o)_o \leq d(o, \omega_\upsilon\, o)$ for $n' \geq 0$ and $n \geq \upsilon(\check{\omega}, \omega)$, and $(\check{\omega}_{n'} o, \omega_n\, o)_o \leq d(o, \omega_n\, o) \leq D_\upsilon$ for $0 \leq n \leq \upsilon(\check{\omega}, \omega)$. This implies

$$\sup_{n,n'\geq 0} e^{\kappa(\check{\omega}_{n'}o,\omega_n\, o)_o} \leq e^{\kappa D_\upsilon} \leq \sum_{i=0}^{\infty} e^{\kappa D_i} 1_{i<\upsilon}.$$

Let us estimate the expectation of the summand. Fixing $\omega = (\check{g}_1, \check{g}_2, \ldots)$ and g_{i+1}, we observe

$$\begin{aligned}\mathbb{E}\left[e^{\kappa D_i} 1_{i<\upsilon}\right] &= \mathbb{E}\left[e^{\kappa D_i} 1_{D_i<c} 1_{i<\upsilon}\right] + \mathbb{E}\left[e^{\kappa D_i} 1_{D_i\geq c} 1_{i<\upsilon}\right] \\ &\leq \mathbb{E}\left[e^{\kappa c} 1_{i<\upsilon}\right] + \mathbb{E}\left[e^{(A+1)\kappa D_i} e^{-A\kappa c}\right] \\ &\leq e^{\kappa c} \cdot K_3 e^{-\kappa_3 i} + e^{-A\kappa c}\, \mathbb{E}_\mu \left[e^{(A+1)\kappa d(o,go)}\right]^i.\end{aligned} \tag{5.3.3}$$

By the assumption, $\mathbb{E}[e^{md(o,go)}] < M$ for some $m, M > 0$. We first take $c = c_1 i$ for each i, where c_1 is large enough so that $e^{c_1 m} \geq M^4$. We then take κ small enough so that $11\kappa < m$ and $\kappa c_1 < \kappa_3/4$, and $(A+1)\kappa = m$. Then the RHS of Inequality 5.3.3 decays exponentially as desired. $\square$

Having established the deviation inequalities, we now observe their consequences.

Chapter 6. Central limit theorem and geodesic tracking

In this chapter, we prove the part of Theorem C regarding displacement. We begin with the following proposition.

6.1 Central limit theorem

Proposition 6.1.1. *Let ω be the random walk on $\mathrm{Mod}(\Sigma)$ generated by a non-elementary measure μ. If μ has finite second moment, then there exists a Gaussian law with variance $\sigma(\mu)^2$ to which $\frac{1}{\sqrt{n}}(d(o, \omega_n\, o) - n\lambda)$ converges in law.*

Proof. Since μ has finite second moment, Proposition 5.3.1 gives the uniform fourth-moment deviation inequality. Then Theorem 4.2 of [MS20] asserts that $[d(o, \omega_n\, o) - \lambda n]/\sqrt{n}$ converges to a Gaussian law in distribution. For completeness, we explain this result in detail.

We first fix $M > 0$ and consider the random variables

$$Y_{k,i} = d(\omega_{2^k M(i-1)}\, o, \omega_{2^k Mi}\, o), \quad b_{k,i} = (\omega_{2^k M(i-1)}\, o, \omega_{2^k M(i+1)}\, o)_{\omega_{2^k Mi}\, o}$$

(see Figure 6.1) and their balanced versions

$$\bar{Y}_{k,i} = Y_{k,i} - \mathbb{E}[Y_{k,i}], \quad \bar{b}_{k,i} = b_{k,i} - \mathbb{E}[b_{k,i}].$$

Observe the following:

1. each of $\{Y_{k,i}\}_{i \in \mathbb{Z}}$, $\{b_{k,i}\}_{i \in 2\mathbb{Z}+1}$, $\{b_{k,i}\}_{i \in 2\mathbb{Z}}$ is a family of i.i.d;
2. there exists $K > 0$ such that $\mathbb{E}[b_{k,i}^2] < K$;
3. $\mathbb{E}[\bar{b}_{k,i}^2] \le \mathbb{E}(|b_{k,i}| + \mathbb{E}\,|b_{k,i}|)^2 \le 4\,\mathbb{E}[b_{k,i}^2] \le 4K$, and
4. $Y_{k+1,i} = Y_{k,2i-1} + Y_{k,2i} - 2b_{k,2i-1}$ for each k, i.

We first show that $\frac{1}{\sqrt{n}}[\mathbb{E}[d(o, \omega_n\, o)] - n\lambda] \to 0$ as $n \to \infty$. Observe that

$$\frac{1}{2^k M}\,\mathbb{E}[Y_{k,1}] = \frac{1}{2^k M}\sum_{i=1}^{2^k}\mathbb{E}[Y_{0,i}] - \frac{2}{2^k M}\sum_{t=0}^{k-1}\left[\sum_{i=1}^{2^{k-t-1}}\mathbb{E}[b_{t,2i-1}]\right].$$

The LHS converges to the escape rate λ as $k \to \infty$, and the first term of the RHS is always $\frac{1}{M}\,\mathbb{E}[d(o, \omega_M\, o)]$. Finally, since $\mathbb{E}[b_{t,2i-1}] < \sqrt{K}$ for any t and i, the second term of the RHS is bounded by $2\sqrt{K}/M$. Hence we deduce $|\sqrt{n}\lambda - \frac{1}{\sqrt{n}}\,\mathbb{E}[d(o, \omega_n\, o)]| \le 2\sqrt{K}/\sqrt{n}$ as desired.

From now on we take $M = 2^m$ for positive integers m. Observe that

$$\frac{1}{\sqrt{2^{k+m}}} Y_{k,1} = \frac{1}{\sqrt{2^{k+m}}}\sum_{i=1}^{2^k} Y_{0,i} - \frac{2}{\sqrt{2^{k+m}}}\sum_{t=0}^{k-1}\left[\sum_{i=1}^{2^{k-t-1}} b_{t,2i-1}\right]. \tag{6.1.1}$$

The same type of identity holds for balanced versions also.

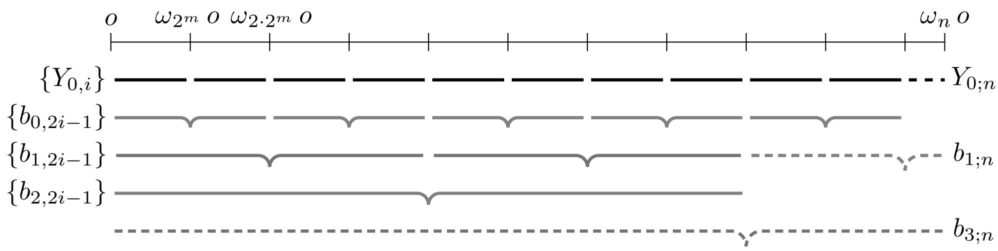

Figure 6.1: $\{Y_{k,i}\}$, $\{Y_{k;n}\}$, $\{b_{k,i}\}$ and $\{b_{k;n}\}$ for $10\cdot 2^m \le n \le 11\cdot 2^m$. Here $b_{0;n} = b_{2;n} = 0$ since $2^m(2\lfloor n/2^{m+1}\rfloor + 1) = 11\cdot 2^m \ge n$ and $2^{m+2}(2\lfloor n/2^{m+3}\rfloor + 1) = 12\cdot 2^m \ge n$.

Let us investigate the error term $\sum_t \sum_i \bar{b}_{t,2i-1}$. For each t, $\sum_i \bar{b}_{t,2i-1}/\sqrt{2^{k+m}}$ is the sum of 2^{k-t-1} independent variables, each of whose variance is bounded by $K/2^{k+m}$. Thus, this sum has variance less than $K/2^{m+t+1}$ and

$$\mathbb{P}\left(E_t := \left\{\left|\frac{1}{\sqrt{2^{k+m}}}\sum_{i=1}^{2^{k-t-1}} \bar{b}_{t,2i-1}\right| \ge 2^{-m/3}2^{-t/4}\right\}\right) \le \frac{K}{2^{m/3+t/2+1}}$$

holds by Chebyshev. Thus, $\frac{1}{\sqrt{2^{k+m}}}\sum_t\sum_i \bar{b}_{t,2i-1}$ is bounded by $7\cdot 2^{-m/3}$ outside $\cup_t E_t$, where $\mathbb{P}(\cup_t E_t) \le 8K\cdot 2^{-m/3}$.

Meanwhile, by the classical CLT, $\frac{1}{\sqrt{2^{k+m}}}\sum_{i=1}^{2^k} \bar{Y}_{0,i}$ converges to a Gaussian law $\mathscr{N}(0,\sigma_m)$ as k increases. Hence, the random variables $\frac{1}{\sqrt{2^k}}[d(o,\omega_{2^k}\, o) - \mathbb{E}[d(o,\omega_{2^k}\, o)]]$ are eventually $(16K+15)\cdot 2^{-m/3}$-close to $\mathscr{N}(0,\sigma_m)$ in the Lévy metric. This implies that $\mathscr{N}(0,\sigma_m)$ are Cauchy, they converge to a Gaussian law $\mathscr{N}(0,\sigma)$ (and $\lim_m \sigma_m = \sigma$).

To deal with distributions at general steps, we consider auxiliary variables

$$\begin{aligned} Y_{k;n} &= d(\omega_{2^{k+m}\lfloor n/2^{k+m}\rfloor}\, o, \omega_n\, o),\\ b_{k;n} &= \begin{cases} (\omega_{2^{k+m+1}\lfloor n/2^{m+k+1}\rfloor}\, o, \omega_n\, o)_{\omega_{2^{k+m}(2\lfloor n/2^{m+k+1}\rfloor+1)}\, o} & \text{if } 2^{k+m}(2\lfloor n/2^{m+k+1}\rfloor + 1) < n\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Here, $\mathbb{E}[b_{k;n}^2] \le 4K$ still holds for any k and n. We now realize that

$$\begin{aligned} &\frac{1}{\sqrt{n}}[d(o,\omega_n\, o) - \mathbb{E}[d(o,\omega_n\, o)]]\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor n/2^m\rfloor} \bar{Y}_{0,i} + \frac{1}{\sqrt{n}}\bar{Y}_{0;n} - \frac{2}{\sqrt{n}}\sum_{2^{m+t}\le n}\left[\bar{b}_{t;n} + \sum_{i=1}^{\lfloor n/2^{m+t+1}\rfloor}\bar{b}_{t,2i-1}\right]. \end{aligned} \tag{6.1.2}$$

As $n\to\infty$, the first term converges to $\mathscr{N}(0,\sigma_m)$ in law. The second term converges to 0 in probability, and in fact, almost surely. This is because finitely many laws $\{Y_{0,1;i} : i = 0,\ldots,2^m-1\}$ have finite variances. Moreover, for $2^{m+t}\le n$ we have

$$Var\left(\frac{1}{\sqrt{n}}\left[b_{t;n} + \sum_{i=1}^{\lfloor n/2^{m+t+1}\rfloor} b_{t,2i-1}\right]\right) \le \frac{4K}{n}\cdot\left[\left\lfloor\frac{n}{2^{m+t+1}}\right\rfloor + 1\right] \le \frac{4K}{2^{m+t}}.$$

This implies that the final term is bounded by $7\cdot 2^{-m/3}$ outside an event with probability at most $16K\cdot 2^{-m/3}$. In conclusion, $\frac{1}{\sqrt{n}}[d(o,\omega_n\, o) - \mathbb{E}[d(o,\omega_n\, o)]]$ is eventually $(32K+15)2^{-m/3}$-close to $\mathscr{N}(0,\sigma_m)$ for each m. Since $\mathscr{N}(0,\sigma_m)\to\mathscr{N}(0,\sigma)$, we conclude $\frac{1}{\sqrt{n}}[d(o,\omega_n\, o) - \mathbb{E}[d(o,\omega_n\, o)]] \to \mathscr{N}(0,\sigma)$. $\square$

Next, we establish the necessary and sufficient condition for the limiting Gaussian distribution to be non-degenerate.

Proposition 6.1.2. *Let ω be the random walk on* $\mathrm{Mod}(\Sigma)$ *generated by a non-elementary measure μ with finite second moment, and let $\mathcal{N}(0, \sigma(\mu))$ be the Gaussian law to which $\frac{1}{\sqrt{n}}(d(o, \omega_n\, o) - n\lambda)$ converges in law. Then $\sigma(\mu)$ is strictly positive if and only if μ is non-arithmetic.*

Proof. First assume that μ is non-arithmetic. Then there exists $g, h \in \operatorname{supp} \mu^{*M}$ that has distinct translation lengths. By taking powers if necessary, we may assume that $d(o, go) - d(o, ho) \geq 104E_0$. Let $\eta, \eta' \in (\operatorname{supp} \mu)^M$ be the sequences with $\Pi(\eta) = g$ and $\Pi(\eta') = h$.

Note that at least $N_0 - 2$ choices $s' \in S$ satisfies that $(g^{-1}o, \Gamma(s'))$ and $(h^{-1}o, \Gamma(s'))$ are K_0-aligned. We gather $N_0/3$ choices out of them to define a Schottky subset $S_2 \subseteq S$. Now, we consider the condition for $s \in S$ that:

$$\left(g\Pi(s')o, \Gamma^{-1}(s),\right), \left(h\Pi(s')o, \Gamma^{-1}(s)\right)$$

are K_0-aligned for all $s' \in S_2$. These are $2N_0/3$ alignment conditions, and by the property of Schottky set S, there exist at least $N_0/3$ elements of $s \in S$ that satisfy the above condition. We gather them and name $S_1 \subseteq S$.

We now consider the decomposition

$$\mu^{4M_0+M} = \alpha \left(\mu_{S_1}^2 \times (1/2_{\{\eta\}} + 1/2_{\{\eta'\}}) \times \mu_{S_2}^2\right) + (1-\alpha)\nu$$

for some $0 < \alpha < 1$ and a probability measure ν. This is the first model described in Section 4.2, and we have

$$\mathbb{P}(\#\mathcal{P}_n(\omega) \leq Kn) \leq Ke^{-n/K}$$

for some $K > 0$. Let $\mathcal{E}$ be an equivalence relation made by v-pivoting at the first 2^m pivotal times till step n, where $m = \lfloor \log_2 Kn \rfloor$. We observe:

Claim 6.1.3. $Var\left[d(o, \omega_n) \,\middle|\, \mathcal{E}\right] \geq 900E_0^2 2^m \geq 450E_0^2 Kn.$

Proof of Claim 6.1.3. Let $\mathcal{P}(\mathcal{E}) = \{i(1) < \ldots < i(2^m) < \ldots\}$. We define

$$x_{2l-1} := \omega_{i(l)+2M_0}\, o, \;\; x_{2l} := \omega_{i(l)+2M_0+M}\, o \quad (l = 1, \ldots, 2^m - 1)$$

and

$$x_{2^{m+1}-1} := \omega_{i(2^m)+2M_0}\, o, \quad x_{2^{m+1}} := \omega_n\, o, \quad x_0 := o.$$

We then have $(x_i, x_k)_{x_j} < E_0$ for all $i < j < k$ due to Lemma 4.1.1 and Proposition 3.1.4. Moreover, $d(x_{2l-2}, x_{2l-1})$ is always fixed and $\{d(x_{2l-1}, x_{2l})\}_{l=1}^{2^m-1}$ is the collection of $2^m - 1$ independent RVs that have value $d(o, go)$ for probability $1/2$ and $d(o, ho)$ for probability $1/2$.

We will inductively prove that

$$Var\left[d(x_{2^k(l-1)}, x_{2^k l}) \,\middle|\, \mathcal{E}\right] \geq E_0^2 \left[900 \cdot 2^k + 240 \cdot 2^{k/2}\right].$$

for $k = 1, \ldots, m+1$ and $l = 1, \ldots, 2^{m-k+1}$. The claim follows when k arrives at $m+1$ and the conditional variances are summed up.

Let us consider the case $k = 1$. The value $d(x_{2l-2}, x_{2l})$ only depends on the choice of $(g_{i(l)+2M_0+1}, \ldots, g_{i(l)+2M_0+M})$ between η and η', each for probability $1/2$. If we let

$$w = \begin{cases} (\omega_{i(l-1)+2M_0+M})^{-1}\, \omega_{i(l)+2M_0} = g_{i(l-1)+2M_0+M+1} \cdots g_{i(l)+2M_0} & 1 < l \leq 2^m, \\ \omega_{i(1)+2M_0} = g_1 \cdots g_{i(1)+2M_0} & l = 1, \end{cases}$$

we have

$$
\begin{aligned}
Var[d(x'_{2l-2}, x'_{2l})] &= \left[\frac{1}{2}|d(o, wgo) - d(o, who)|\right]^2 \\
&= \frac{1}{4}\Big|[d(o, wo) + d(o, go) - 2(o, wgo)_{wo}] - [d(o, wo) + d(o, ho) - 2(o, who)_{wo}]\Big|^2 \\
&\geq \frac{1}{4}\left(|d(o, go) - d(o, ho)| - 2E_0\right)^2 \\
&\geq 2500E_0^2 \geq E_0^2 \cdot \left[1800 + 240\sqrt{2}\right]
\end{aligned}
$$

for $l = 1, \ldots, 2^m - 1$. For $l = 2^m$ we can obtain the conclusion in a similar way by also considering $w' := g_{i(l)+2M_0+M+1} \cdots g_n$.

Suppose now that $Y_1 = d(x_{2^k(2l-2)}, x_{2^k(2l-1)})$ and $Y_2 = d(x_{2^k(2l-1)}, x_{2^k \cdot 2l})$ satisfy the estimation for some $1 \leq k \leq m$ and $1 \leq l \leq 2^{m-k}$. We now estimate the variance of $Y = d(x_{2^{k+1}(l-1)}, x_{2^{k+1}l}) = Y_1 + Y_2 - b$, where $b = 2(x_{2^k(l-2)}, x_{2^k l})_{x_{2^k(l-1)}}$. Since Y_1, Y_2 are independent and $0 \leq b \leq 2E_0$,

$$
\begin{aligned}
Var(Y) &\geq Var(Y_1) + Var(Y_2) - 2E_0 \cdot \sqrt{Var(Y_1)} - 2E_0 \cdot \sqrt{Var(Y_2)} \\
&= Var(Y_1)\left[1 - \frac{2E_0}{\sqrt{Var(Y_1)}}\right] + Var(Y_2)\left[1 - \frac{2E_0}{\sqrt{Var(Y_2)}}\right] \\
&\geq 2 \cdot E_0^2\left[900 \cdot 2^k + 240 \cdot 2^{k/2}\right]\left[1 - \frac{2E_0}{E_0 \cdot 30 \cdot 2^{k/2}}\right] \\
&\geq 2 \cdot E_0^2\left[900 \cdot 2^k + 180 \cdot 2^{k/2} - 16\right] \\
&\geq E_0^2\left[900 \cdot 2^{k+1} + 240 \cdot 2^{(k+1)/2} + (360 - 240\sqrt{2})2^{k/2} - 16\right]
\end{aligned}
$$

holds. Since $360 - 240\sqrt{2} \geq 16$, we have the desired conclusion for $k + 1$. □

Since the equivalence classes that have more than Kn pivotal times take up probability at least $1 - Ke^{-n/K}$, we have

$$
Var[d(o, \omega_n\, o)] \geq \mathbb{E}[Var[d(o, \omega_n\, o)|\mathcal{E}]] \geq (1 - Ke^{-n/K}) \cdot 450E_0^2 Kn
$$

and $\sigma_m := \frac{1}{\sqrt{2^m}}\sqrt{Var(d(o, \omega_{2^m}\, o))}$ is bounded away from zero. This concludes that $\lim_n \sigma_n = \sigma$ is strictly positive.

Let us now establish the converse direction. Consider the inequality

$$
|d(o, go) - d(o, ho)| \leq 104E_0.
$$

If this holds for all $g, h \in \operatorname{supp} \mu^{*n}$ for all n, then we have

$$
|d(o, go) - \lambda n| \leq 104E_0
$$

for all $g \in \operatorname{supp} \mu^{*n}$ for all n and the limiting distribution will be degenerate. In other words, if the limiting distribution is non-degenerate, then there exist n and $g, h \in \operatorname{supp} \mu^{*n}$ such that $|d(o, go) - d(o, ho)| > 104E_0$. As in the beginning of the proof, we can find $s' \in S$ such that $(g^{-1}o, \Gamma(s'))$ and $(h^{-1}o, \Gamma(s'))$ are K_0-aligned, and then find $s \in S$ such that

$$
\left(g\Pi(s')o, \Gamma^{-1}(s),\right), \left(h\Pi(s')o, \Gamma^{-1}(s)\right)
$$

are K_0-aligned. Then

$$
(\Gamma(s), \Pi(s)g\Gamma(s')), (\Gamma(s), \Pi(s)h\Gamma(s'))
$$

are D_0-aligned by Lemma 3.1.2, and consequently, that

$$(o, g\Gamma(s'), g\Pi(s')\Gamma(s), g\Pi(s')\Pi(s)g\Gamma(s'), \ldots, (g\Pi(s')g)^{n-1}g\Pi(s')\Gamma(s), (g\Pi(s')\Pi(s))^n o),$$
$$(o, g\Gamma(s'), h\Pi(s')\Gamma(s), h\Pi(s')\Pi(s)h\Gamma(s'), \ldots, (h\Pi(s')h)^{n-1}h\Pi(s')\Gamma(s), (h\Pi(s')\Pi(s))^n o)$$

are D_0-aligned. In particular, the Gromov products among the endpoints are bounded by E_0 and we deduce

$$|\tau(g\Pi(s)\Pi(s')) - d(o, go) + d(o, \Pi(s)o) + d(o, \Pi(s')o)]| \le 3E_0,$$
$$|\tau(h\Pi(s)\Pi(s')) - d(o, ho) + d(o, \Pi(s)o) + d(o, \Pi(s')o)]| \le 3E_0.$$

In summary, we have obtained two elements $g\Pi(s)\Pi(s')$, $h\Pi(s)\Pi(s')$ in the support of $\mu^{*(n+2M_0)}$ whose translation lengths are distinct; μ is non-arithmetic. □

6.2 Berry-Esseen type estimates

We now establish a quantitative control of the error term in the CLT.

Theorem 6.2.1. *Let ω be the random walk generated by a non-elementary, non-arithmetic measure μ on G. Suppose that μ has finite third moment, and let $F_n(x)$ be the distribution of $[d(o, \omega_n\, o) - n\lambda]/\sigma\sqrt{n}$. Then there exists $K > 0$ such that*

$$|F_n(x) - \mathcal{N}(x)| \le \frac{K}{\sqrt[5]{n}}$$

holds for all x and n.

Proof. Let us denote $\frac{1}{\sqrt{n}}\sqrt{Var[d(o, \omega_n\, o)]}$ by σ_n. In the proof of Proposition 6.1.1, we proved that:

1. the RVs $\left(\frac{1}{\sqrt{n}} d(o, \omega_n\, o) - \mathbb{E}[d(o, \omega_n\, o)]\right)_{n>0}$ converges to $\mathscr{N}(0, \sigma)$ for some $\sigma > 0$, and
2. for each $k > 0$, the RVs $\{\frac{1}{\sqrt{k2^n}} d(o, \omega_{k2^n}\, o) - \mathbb{E}[d(o, \omega_{k2^n}\, o)]\}_{n>0}$ are eventually $K/\sqrt[3]{k}$-close to $\mathscr{N}(0, \sigma_k)$.

These two imply that $\mathscr{N}(0, \sigma_k)$ and $\mathscr{N}(0, \sigma)$ are $K/\sqrt[3]{k}$-close. Moreover, since we have uniform 6-th moment deviation inequality, we have $\mathbb{E}\,|d(o, \omega_n) - \mathbb{E}[d(o, \omega_n)]|^3 \le K' n^{3/2}$ for some $K' > 0$ ([MS20, Theorem 4.9]).

Given n, we fix the following notations throughout the proof:

$$\begin{aligned}
y_i &:= \omega_i\, o \quad (i = 0, \ldots, n),\\
N_2 &:= \lfloor n^{2/5} \rfloor,\\
N_3 &:= \lfloor n/N_2 \rfloor,\\
Y_{i,n} &:= d(y_{(i-1)N_3}, y_{iN_3}), \quad (i = 1, \ldots, N_2)\\
Y_n^* &:= d(y_{N_2 N_3}, y_n),\\
c^* &:= (o, y_n)_{y_{N_2 N_3}}.
\end{aligned}$$

Next, we define a family of sequences $\{(m(i;k))_{i=0}^{2^k}\}_{k=0}^{\lfloor \log_2 N_2 \rfloor}$ as follows. First we set $m(0;0) = 0$, $m(1;0) = N_2$. Now given $(m(i;k-1))_{i=0}^{2^{k-1}}$ for $k \le \log_2 N_2$, we define $m(2i;k) := m(i;k-1)$ for $i = 0, \ldots, 2^{k-1}$ and

$$m(2i-1;k) := m(i-1;k-1) + \left\lfloor \frac{m(i;k-1) - m(i-1;k-1)}{2} \right\rfloor$$

for $i = 1, \ldots, 2^{k-1}$. Then

$$2^{\lfloor \log_2 N_2 \rfloor - k} \leq m(i;k) - m(i-1;k) \leq 2^{\lfloor \log_2 N_2 \rfloor - k+1} \tag{6.2.1}$$

holds for $k = 0, \ldots, \lfloor \log_2 N_2 \rfloor$ and $i = 1, \ldots, 2^k$.

From this sequences we define

$$b_{i;k} := (y_{N_3 \cdot m(2i-2;k)}, y_{N_3 \cdot m(2i;k)})_{y_{N_3 \cdot m(2i-1;k)}}$$

for $k = 1, \ldots, \lfloor \log_2 N_2 \rfloor - 1$ and $i = 1, \ldots, 2^{k-1}$. Finally, note that

$$\left(m\left(0; \lfloor \log_2 N_2 \rfloor\right), m\left(1; \lfloor \log_2 N_2 \rfloor\right), \ldots, m\left(2^{\lfloor \log_2 N_2 \rfloor}; \lfloor \log_2 N_2 \rfloor\right)\right)$$

is a sequence that increases by 1 or 2 at each step. Let $m'(1) < \ldots < m'(N_2 - 2^{\lfloor \log_2 N_2 \rfloor})$ be the numbers in the sequence that differs with the previous step by 2, and define

$$c_t := (y_{N_3 \cdot m'(t)}, y_{N_3 \cdot m'(t)-2})_{y_{N_3 \cdot m'(t)-1}}.$$

We then observe that

$$\begin{aligned} d(o, \omega_n\, o) &= d(o, \omega_{N_2 N_3}\, o) + d(\omega_{N_2 N_3}\, o, \omega_n\, o) - 2(o, \omega_n\, o)_{\omega_{N_2 N_3}\, o} \\ &= d(o, \omega_{N_2 N_3}\, o) + Y_n^* - 2c^* \\ &= \sum_{i=1}^{N_2} Y_{i,n} + 2\left(\sum_{k=1}^{\lfloor \log_2 N_2 \rfloor} \sum_{i=1}^{2^{k-1}} b_{i;k}\right) + 2\left(\sum_{i=1}^{N_2 - 2^{\lfloor \log_2 N_2 \rfloor}} c_i\right) + Y_n^* - 2c^*. \end{aligned} \tag{6.2.2}$$

For convenience, let us denote by $\bar{Y}$ the centered version $Y - \mathbb{E}[Y]$ of an RV Y. We then also have

$$\begin{aligned} \frac{1}{\sigma\sqrt{n}}[d(o, \omega_n\, o) - \lambda n] = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{N_2} \bar{Y}_{i,n} - \frac{2}{\sigma\sqrt{n}}\left(\sum_{k=1}^{\lfloor \log_2 N_2 \rfloor} \sum_{i=1}^{2^{k-1}} \bar{b}_{i;k}\right) - \frac{2}{\sigma\sqrt{n}}\left(\sum_{i=1}^{N_2 - 2^{\lfloor \log_2 N_2 \rfloor}} \bar{c}_i\right) \\ + \frac{1}{\sigma\sqrt{n}} \bar{Y}_n^* - \frac{2}{\sigma\sqrt{n}} \bar{c}^* + \left(\frac{1}{\sigma\sqrt{n}} \mathbb{E}[d(o, \omega_n\, o)] - \frac{\sqrt{n}\lambda}{\sigma}\right). \end{aligned} \tag{6.2.3}$$

We now deal with each term of Equation 6.2.3. First, note that

$$\mathbb{E}\left[\frac{\sqrt{N_2}}{\sigma\sqrt{n}} \bar{Y}_{i,n}^3\right] \leq K' \frac{N_3^{3/2} N_2^{3/2}}{n^{3/2}} \leq K'$$

and

$$\mathbb{E}\left[\frac{\sqrt{N_2}}{\sigma\sqrt{n}} \bar{Y}_{i,n}^2\right] \geq 0.9 s^2 \frac{N_3 \cdot N_2}{n} \geq 0.8 s^2$$

for large enough n. Then the classical Berry-Esseen estimate asserts that there exists $K > 0$ (that works for all large n) such that

$$|F_n^{(1)}(x) - \mathcal{N}'(x)| \leq K \frac{1}{\sqrt[5]{n}}$$

holds for all $x \in \mathbb{R}$, where $F_n^{(1)}(x)$ is the distribution of $\frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{N_2} \bar{Y}_{i,n}$ and $\mathcal{N}'$ is the distribution of $\mathcal{N}\left(0, (\sigma_{N_3}/\sigma) \cdot \sqrt{(N_2 N_3)/n}\right)$. Since $\mathcal{N}(0, \sigma_{N_3})$ and $\mathcal{N}(0, \sigma)$ are $K/\sqrt[5]{n}$-close, we have

$$|\mathcal{N}'(x) - \mathcal{N}_1(x)| \leq \frac{K}{\sqrt[5]{n}}$$

for all x where $\mathcal{N}_1(x)$ is the distribution of $\mathcal{N}(0, \sqrt{(N_2 N_3)/n})$. Moreover, we note $1 - \sqrt{N_2 N_3/n} \leq K/n^{2/5}$; this implies $|\mathcal{N}'(x) - \mathcal{N}(x)| \leq K/\sqrt[5]{n}$ for all x also. Since $\mathcal{N}(x)$ is Lipschitz, it now suffices to show that the remaining terms are $O(1/\sqrt[5]{n})$ outside a set of probability $O(1/\sqrt[5]{n})$.

To deal with the second summation, let us recall that $\{\bar{b}_{i;k}\}_i$ is a family of independent RVs that have uniformly bounded 6th moment. Hence,

$$\mathbb{E}\left[\sum_{i=1}^{2^{k-1}} \bar{b}_{i;k}\right]^6 \le K(2^{k-1})^3$$

for some K that does not depend on k and n. Using the Chebyshev inequality, we have $|\frac{1}{\sigma\sqrt{n}}\sum_i \bar{b}_{i;k}| < n^{-1/5}2^{-k/6}$ outside a set of probability $O(n^{-9/5}2^{4k})$. Summing up these effects, we have

$$\mathbb{P}\left(\frac{2}{\sigma\sqrt{n}}\left(\sum_{k=1}^{\lfloor \log_2 N_2 \rfloor}\sum_{i=1}^{2^{k-1}} \bar{b}_{i;k}\right) > \frac{1}{\sqrt[5]{n}}\right) \le 2 \cdot 2^{4\log_2 N_2} \cdot O(n^{-9/5}) = O(n^{-1/5}).$$

Similarly, the third term of Equation 6.2.3 has 6th moment of order $O(n^{-9/5})$ and is bounded by $1/\sqrt[5]{n}$ outside a set of probability $O(n^{-3/5})$. Moreover, the fourth term is a sum of at most N_2 independent RVs with uniformly bounded variance, so its variance is bounded by $O(N_2/n) = O(n^{-3/5})$. Again, it is bounded by $1/\sqrt[5]{n}$ outside a set of probability $O(n^{-1/5})$. The fifth term has variance $O(1/n)$ and can be handled similarly.

Finally, recall the proof of Proposition 6.1.1 that the error arising from the average, i.e., $|\sqrt{n}\lambda - \frac{1}{\sqrt{n}}\mathbb{E}[d(o, \omega_n\, o)]|$, is of order $O(1/\sqrt{n})$. This finishes the proof. $\square$

6.3 Converse of the central limit theorem

Proposition 6.3.1. *Let μ be a non-elementary measure on* $\mathrm{Mod}(\Sigma)$ *with infinite second moment. Then for any sequence $(c_n)_n$ of real values, $\{\frac{1}{\sqrt{n}}[d(o, \omega_n\, o) - c_n]\}_n$ does not converge in law.*

Proof. For each pair of subsets S_1, S_2 of S with cardinality $N_0/2$, we define

$$A(S_1, S_2) := \left\{g \in G : (g\,\Pi(s_2)o, \Gamma^{-1}(s_1)) \text{ and } (g^{-1}o, \Gamma(s_2)) \text{ are } K_0\text{-aligned for all } s_1 \in S_1,\ s_2 \in S_2\right\}.$$

Given an element g of G, there exist at least $N_0 - 1$ Schottky choices $s_2 \in S$ that makes $(g^{-1}o, \Gamma(s_2))$ K_0-aligned. Choosing $N_0/2$ choices $s_2^{(1)}, \ldots, s_2^{(N_0/2)}$ among them, we now want $(g\,\Pi(s_2^{(i)})o, \Gamma^{-1}(s_1))$ to be K_0-aligned for each $i = 1, \ldots, N_0/2$: there exist at least $N_0/2$ Schottky choices realizing them. As a result, each $g \in G$ belongs to $A(S_1, S_2)$ for some subsets $S_1, S_2 \in \binom{S}{N_0/2}$. Hence, we have

$$\sum_{\substack{S_1, S_2 \subseteq S \\ \#S' = N_0/2}} \sum_{g \in A(S_1,S_2)} \mu(g) d(o, go)^2 \ge \sum_{g \in G} \mu(g) d(o, go)^2 = +\infty,$$

which implies that

$$\mathbb{E}\left[d(o, go)^2 \,|\, g \in A(S_1, S_2)\right] = +\infty$$

for some $S_1, S_2 \subseteq S$ with cardinality $N_0/2$. Let μ_{S_1} and μ_{S_2} be the uniform measure on S_1 and S_2, respectively, and

$$\mu' := \begin{cases} \mu(g)/\mu(A(S_1, S_2)) & g \in A(S_1, S_2) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbb{E}_{\mu'}[d(o, go)^2] = +\infty$ and $\mu' \le \frac{1}{\mu(A(S_1,S_2))}\mu$ hold. We now consider the decomposition

$$\mu^{(4M_0+1)} = \alpha(\mu_{S_1}^2 \times \mu' \times \mu_{S_2}^2) + (1 - \alpha)\nu$$

for some $0 < \alpha < 1$ and ν. We employ the first model described in Section 4.2 and deduce that

$$\mathbb{P}(\#\mathcal{P}_n(\omega) \le Kn) \le Ke^{-n/K}$$

for some $K > 0$.

We now gather all sample paths with at least 2^m pivotal times till n, where $m = \lfloor \log_2 Kn \rfloor$. By venti-pivoting at the first $2^m - 1$ pivotal times, these sample paths are grouped into equivalence classes; let $\mathcal{E}$ be such an equivalence class. As in the proof of Proposition 6.1.2, we label $\mathcal{P}(\mathcal{E}) = \{i(1) < \ldots < i(2^m) < \ldots\}$ and define $x_0, \ldots, x_{2^{m+1}}$. Then the following hold.

1. $\{d(x_{2l-2}, x_{2l-1})\}_{l=1}^{2^m}$, $d(x_{2^{m+1}-1}, x_{2^m})$ are uniform in the equivalence class $\mathcal{E}$.
2. $\{d(x_{2l-1}, x_{2l})\}_{l=1}^{2^m-1}$ are i.i.d. with infinite second moment.
3. For any $i < j < k$, x_i and x_k are endpoints of a D_0-aligned sequence of Schottky segments, one of whose endpoint is x_j. By Proposition 3.1.5, we have $(x_i, x_k)_{x_j} < E_0$ always.
4. For any $i < j < k \le i' < j' < k'$, $(x_i, x_k)_{x_j}$ and $(x_{i'}, x_{k'})_{x_{j'}}$ are independent.

Now observe the equality

$$d(o, \omega_n\, o) = \underbrace{\sum_{i=1}^{2^m} d(x_{2i-2}, x_{2i-1}) + d(x_{2^{m+1}-1}, x_{2^{m+1}})}_{I_1} + \underbrace{\sum_{i=1}^{2^m-1} d(x_{2i-1}, x_{2i})}_{I_2} \underbrace{-2\sum_{l=0}^{m}\sum_{k=1}^{2^{m-l}} (x_{2^l(2k-2)}, x_{2^l\cdot 2k})_{x_{2^l(2k-1)}}}_{I_3}.$$

The third term I_3 is composed of sums of 2^{m-l} independent RVs bounded by E_0. Using the estimation of the variance and Chebyshev's inequality, one can deduce that

$$\mathbb{P}\left(I_3 - \mathbb{E}[I_3 \,|\, \mathcal{E}] > 800E_0 \cdot 2^{m/2}\right) \le 1/2000.$$

Meanwhile, I_1 is constant on $\mathcal{E}$.

At the moment, we consider another random walk $\check{\omega}$, independent from ω but with the same distribution with ω. We copy the exact same procedure to pick an independent equivalence class $\dot{\mathcal{E}}$ and define $\dot{x}_l$'s, $\dot{I}_1$, $\dot{I}_2$ and $\dot{I}_3$.

We now compare $I_1 + \mathbb{E}[I_3 \,|\, \mathcal{E}]$ and $\dot{I}_1 + \mathbb{E}[\dot{I}_3 \,|\, \dot{\mathcal{E}}]$. Since the situation is symmetric, the former will win or tie with the latter for probability at least 0.5. We fix a combination $(\mathcal{E}, \dot{\mathcal{E}})$ falling into this case and compare I_2 and $\dot{I}_2$. Since $I_2 - \dot{I}_2$ is the sum of $2^m - 1$ i.i.d.s $\{d(x_{2i-1}, x_{2i}) - d(\dot{x}_{2i-1}, \dot{x}_{2i})\}_i$ of *symmetric* distribution with infinite second moment, for any $K' > 0$ we have

$$\mathbb{P}(I_2 - \dot{I}_2 \ge K' 2^{m/2}) \ge 3/40$$

for sufficiently large m. We briefly explain this well-known trick (e.g., in Exercise 3.4.3, [Dur19]) for the sake of completeness. We truncate the RV $Y_i := d(x_{2i-1}, x_{2i}) - d(\dot{x}_{2i-1}, \dot{x}_{2i})$ into two parts, $U_i = Y_i 1_{|Y_i| \le M}$ and $V_i = 1_{|Y_i| > M}$ for some large threshold M such that $Var(U_i) \ge K'^2$. Since Y_i has infinite second moment, such a threshold always exists. Now, we note that $\sum_{i=1}^{2^m-1} Y_i$ is greater than $K'\sqrt{n}$ if $\sum_{i=1}^{2^m-1} U_i$ is so and $\sum_{i=1}^{2^m-1} V_i$ is nonnegative. Since the parity of V_i is independent of U_i, $\sum_i V_i$ is nonegative for probability at least $1/2$ for any prior combination of U_i's. Hence, we obtain

$$\mathbb{P}\left(\sum_{i=1}^{2^m-1} Y_i \ge K'\sqrt{2^m}\right) \ge 0.5\, \mathbb{P}\left(\sum_{i=1}^{2^m-1} U_i \ge K'\sqrt{2^m}\right).$$

Here, we can apply CLT on U_i since it has finite variance:

$$\begin{aligned}
&\lim_{m\to\infty} \mathbb{P}\left(\sum_{i=1}^{2^m-1} U_i \geq K' 2^{m/2}\right) \\
&= \lim_{m\to\infty} \mathbb{P}\left(\frac{1}{\sqrt{2^m-1}\sqrt{Var(U_i)}} \sum_{i=1}^{2^m-1} U_i \geq \frac{K'}{\sqrt{Var(U_i)}} \cdot \sqrt{\frac{2^m}{2^m-1}}\right) \\
&\geq \mathbb{P}(\chi \geq 1) \geq 0.15,
\end{aligned}$$

where χ has the standard normal distribution.

Combining all these, for arbitrary $K' > 0$, $d(o, \omega_n o) - d(o, \dot{\omega}_n o) \geq 0.5K' 2^{m/2} \geq 0.25K'\sqrt{n}$ for probability at least $(1 - 2Ke^{-n/K} - 2/1000) \cdot (3/40) \geq 1/20$ for sufficiently large n. However, this cannot happen for arbitrary $K' > 0$ if $\frac{1}{\sqrt{n}}[d(o, \omega_n o) - d(o, \dot{\omega}_n o)]$ converged in law. Hence, $\frac{1}{\sqrt{n}} d(o, \omega_n o)$ cannot converge in law even after suitable translation. □

6.4 Law of the iterated logarithm

Throughout this section we set

$$LLn := \begin{cases} \log\log n & n \geq 3 \\ 1 & n < 2, \end{cases} \qquad \alpha(n) := (2nLLn)^{1/2}, \quad \beta(n) := (n/LLn)^{1/2}.$$

In this section, we adapt de Acosta's argument for the classical LIL in [dA83] to prove our LIL. Let us briefly summarize de Acosta's strategy before entering the proof. Let $\{X_i\}$ be a sequence of balanced i.i.d. with $Var(X_i) < K$. In order to investigate the deviation of $\sum_{i=1}^n X_i$ in the order of $\alpha(n)$, de Acosta first truncated X_n to obtain $Y_n := X_n 1_{\{|X_n| \leq \beta(n)\}}$, $Z_n := X_n 1_{\{|X_n| > \beta(n)\}}$ (assume $\mathbb{E}[Y_n] = 0$ at the moment for convenience).

The truncation threshold $\tau\beta(n)$ is so designed that the a.e. convergence of $\sum_{i=1}^n |Z_i|/\alpha(i)$ follows from finite variances of X_i. Kronecker's lemma then implies that the term $(\sum_{i=1}^n Z_i)/\alpha(n)$ does not contribute significantly. For Y_n, we make use of the independence of Y_n, truncation bounds of Y_n and Chebyshev's inequality to deduce

$$\mathbb{P}\left\{\sum_{i=1}^n Y_i/\alpha(n) > t\right\} \leq \exp\left[-\lambda t + \frac{\lambda^2 K}{4LLn} \exp\left(\frac{\lambda}{\sqrt{2LLn}}\right)\right]$$

for any $t, \lambda > 0$. The final trick is to couple the sequence of events $E_n := \{\sum_{i=1}^n X_i/\alpha(n) > t\}$ with a geometric subsequence $E_{\lfloor p^k \rfloor}$, in the sense that

$$\mathbb{P}\left(\cup_{n \geq p^{k_0}} E_n\right) \leq C \sum_{k \geq k_0} \mathbb{P}\left(E_{\lfloor p^k \rfloor}\right). \tag{6.4.1}$$

Choosing suitable t and λ, one can make this series convergent and Borel-Cantelli leads to the a.e. upper bound of $\limsup(\sum_{i=1}^n X_i)/\alpha(n)$.

Let us now return to our setting. The second term in Equation 6.1.2 still converges to 0 when the denominator is replaced with $\alpha(n)$. It is the final term in Equation 6.1.2 that requires de Acosta's argument. The additional obstacle here is that we deal with the infinite sequence $\{\sum_i \bar{b}_{t,2i-1}\}_t$ of sums of i.i.d.; we should not only establish a bound on RHS of Inequality 6.4.1 for each family $\{\bar{b}_{t,2i-1}\}_i$, but also that the bound is summable for t.

Claim 6.4.1. *For any $K' > 0$, there exists $T > 0$ such that*

$$\mathbb{P}\left\{\limsup_n \frac{1}{\alpha(n)}\left|\sum_{t\geq T}\sum_{i=1}^{\lfloor n/2^{m+t+1}\rfloor}\bar{b}_{t,2i-1}\right| > K'\right\} \leq K'.$$

Proof. Let us consider

$$E_{t,i} := \left\{\omega : |\bar{b}_{t,2i-1}| > \beta(2^{t+m+1}i)/2^{t/4}\right\},$$
$$B_{t,2i-1} := \bar{b}_{t,2i-1}1_{E_{t,i}},\ \ B'_{t,2i-1} := \bar{b}_{t,2i-1}1_{E^c_{t,i}},\ \ \bar{B}'_{t,2i-1} := B'_{t,2i-1} - \mathbb{E}\,B'_{t,2i-1}.$$

Note that

$$\begin{aligned}\left|\mathbb{E}\,B'_{t,2i-1}\right| &= |\mathbb{E}\,B_{t,2i-1}| \leq \mathbb{E}\,|B_{t,2i-1}| \leq \mathbb{E}\,|\bar{b}_{t,2i-1}| \\ &= \mathbb{E}\,|b_{t,2i-1} - (\mathbb{E}\,b_{t,2i-1})| \leq 2\,\mathbb{E}\,|b_{t,2i-1}| \leq 2\sqrt{K}, \\ |\bar{B}'_{t,2i-1}| &\leq |B'_{t,2i-1}| + |\mathbb{E}\,B'_{t,2i-1}| \leq 2\cdot\beta(2^{t+m+1}i)/2^{t/4}, \\ \mathbb{E}(\bar{B}'_{t,2i-1})^2 &\leq \mathbb{E}(|B'_{t,2i-1}| + \mathbb{E}\,|B'_{t,2i-1}|)^2 \\ &\leq 4\,\mathbb{E}\,|B'_{t,2i-1}|^2 \leq 4\,\mathbb{E}\,\bar{b}^2_{t,2i-1} \leq 16K.\end{aligned}$$

Our first aim is to show

$$\sum_{t\geq T}\sum_{i=1}^{\infty}\mathbb{E}\,|B_{t,2i-1}|/\alpha(2^{t+m+1}i) < \infty. \tag{6.4.2}$$

Given this, Kronecker's lemma will then imply that

$$\lim_n \frac{1}{\alpha(n)}\left|\sum_{t\geq T}\sum_{i=1}^{\lfloor n/2^{m+t+1}\rfloor}B_{t,2i-1}\right| = 0 \quad \text{a.s.} \tag{6.4.3}$$

In order to show Inequality 6.4.2, we observe

$$\begin{aligned}&\sum_{i=1}^{\infty}\mathbb{E}\,|B_{t,2i-1}|/\alpha(2^{t+m+1}i) \\ &\leq \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{\alpha(2^{t+m+1}i)}\frac{\beta(2^{t+m+1}(i+k+1))}{2^{t/4}}\,\mathbb{P}\left[\frac{\beta(2^{t+m+1}(i+k))}{2^{t/4}} < |\bar{b}_{t,2i-1}| \leq \frac{\beta(2^{t+m+1}(i+k+1))}{2^{t/4}}\right] \\ &= \sum_{j=1}^{\infty}\frac{\beta(2^{t+m+1}(j+1))}{2^{t/4}}\,\mathbb{P}\left[\frac{\beta(2^{t+m+1}j)}{2^{t/4}} < |\bar{b}_{t,1}| \leq \frac{\beta(2^{t+m+1}(j+1))}{2^{t/4}}\right]\cdot\sum_{i=1}^{j}\frac{1}{\alpha(2^{t+m+1}i)}\end{aligned} \tag{6.4.4}$$

for sufficiently large t. Here are used the facts that $\beta(x)$ is increasing for $x \geq 8$ and that $\{\bar{b}_{t,2i-1}\}_i$ are i.i.d. Moreover, we have

$$\sum_{i=1}^{j}\frac{1}{\alpha(2^{t+m+1}i)} \leq \frac{10}{2^{t+m+1}}\beta(2^{t+m+1}j), \quad \beta(2^{t+m+1}(j+1)) \leq 1.1\beta(2^{t+m+1}j)$$

for each j. Hence the last quantity in Inequality 6.4.4 is bounded by

$$\begin{aligned}&11\sum_{j=1}^{\infty}2^{-5t/4-m-1}\beta^2(2^{t+m+1}j)\,\mathbb{P}\left[\frac{\beta(2^{t+m+1}j)}{2^{t/4}} < |\bar{b}_{t,1}| \leq \frac{\beta(2^{t+m+1}(j+1))}{2^{t/4}}\right] \\ &\leq 11\cdot 2^{-3t/4}Var(\bar{b}_{t,1}) \leq 44K\cdot 2^{-3t/4},\end{aligned}$$

which is clearly summable. Note that Inequality 6.4.2 also implies

$$\sum_{t\geq T}\sum_i|\mathbb{E}\,B'_{t,2i-1}|/\alpha(2^{t+m+1}i) = \sum_{t\geq T}\sum_i\mathbb{E}\,|B_{t,2i-1}|/\alpha(2^{t+m+1}i) < \infty.$$

Again by Kronecker's lemma, this implies that

$$\lim_n \frac{1}{\alpha(n)} \left| \sum_{t \geq T} \sum_{i=1}^{\lfloor n/2^{m+t+1} \rfloor} \mathbb{E}\, B'_{t,2i-1} \right| = 0. \tag{6.4.5}$$

We now handle $\{\bar{B}'_{t,2i-1}\}_i$ for $t \geq T$. Since these are balanced i.i.d. with

$$\mathbb{E}(\bar{B}'_{t,2i-1})^2 \leq 16K \quad \text{and} \quad |\bar{B}'_{t,2i-1}| \leq 2^{1-t/4} \cdot \beta(2^{t+m+1} i),$$

we can apply the proof of Lemma 2.2 of [dA83] and deduce that

$$\mathbb{P}\left\{ \sum_{i=1}^{n} \bar{B}'_{t,2i-1} > 2^{-t/8} \sqrt{K} \alpha(2^{m+t+1} n) \right\} \leq \exp\left[-\left(2 \cdot 2^{t/8} - \frac{16}{2^{t/2+m+1}} e^{2\sqrt{2}/\sqrt{K}} \right) LL(2^{m+t+1} n) \right], \tag{6.4.6}$$

which is bounded by $\exp[-2^{t/8} LL(2^{t+m+1} n)]$ for sufficiently large t. Note also that Chebyshev's inequality also implies

$$\mathbb{P}\left[\left| \sum_{i=n}^{2^k} \bar{B}'_{t,2i-1} \right| \geq 2^{-t/8} \sqrt{K} \alpha(2^{m+t+1} \cdot 2^k) \right] \leq \frac{16 \cdot 2^k K}{2^{-t/4} \alpha^2 (2^{m+t+1} \cdot 2^k) K} \leq 1/2 \tag{6.4.7}$$

for any $k \geq 1$ and $n \leq 2^k$.

We now estimate the probability that $|\sum_{i=1}^{\lfloor n/2^{m+t+1} \rfloor} \bar{B}'_{t,2i-1}| > 3 \cdot 2^{-t/8} \sqrt{K} \alpha(n)$ occurs for at least one n. This is bounded by

$$\sum_{k=0}^{\infty} \mathbb{P}\left[\max_{2^k \leq n < 2^{k+1}} \left| \sum_{i=1}^{n} \bar{B}'_{t,2i-1} \right| > 3 \cdot 2^{-t/8} \sqrt{K} \alpha(2^{m+t+1} \cdot 2^k) \right] .$$

By Inequality 6.4.7 and Ottaviani's inequality, this is bounded by

$$2 \sum_{k=0}^{\infty} \mathbb{P}\left[\sum_{i=1}^{2^{k+1}} |\bar{B}'_{t,2i-1}| > 2 \cdot 2^{-t/8} \sqrt{K} \alpha(2^{m+t+1} \cdot 2^k) \right].$$

Since $2\alpha(2^{m+t+1} \cdot 2^k) \geq \alpha(2^{m+t+1} \cdot 2^{k+1})$ for sufficiently large t and all k, we can rely on Inequality 6.4.6 to bound this with

$$2 \sum_{k=0}^{\infty} ([k+m+t+2] \log 2)^{-2 \cdot 2^{t/8}} \leq 2 \sum_{k=t}^{\infty} (k \log 2)^{-4} \leq \frac{1}{t^3 (\log 2)^4}$$

Taking T large enough, we have $\sum_{t \geq T} t^{-3} (\log 2)^{-4} < K'$. Outside this event, we have

$$\frac{1}{\alpha(n)} \sum_{t \geq T} \sum_{i=1}^{\lfloor n/2^{m+t+1} \rfloor} \bar{B}'_{t,2i-1} \leq 3\sqrt{K} \sum_{t \geq T} 2^{-t/8} \leq 30\sqrt{K} \cdot 2^{-T/8} < K' \tag{6.4.8}$$

for all n, once again by taking T large enough. Combining this with Equation 6.4.3 and 6.4.5 yields the conclusion. □

We should also cope with the remaining terms $\bar{b}_{t;n}$'s: note that for each t, only one copy of $\bar{b}_{t;n}$ arises at step n. This forces us to handle each deviation event $\{\bar{b}_{t;n} > K'\alpha(n)\}$ separately (for example, it is hard to rely on Ottaviani's inequality to reduce to subsequential events).

Claim 6.4.2.

$$\limsup_n \frac{1}{\alpha(n)} \left| \sum_{2^{m+t} \leq n} \bar{b}_{t;n} \right| = 0 \quad a.s.$$

Proof. Let $K' > 0$. Given $t \geq 0$ and $1 \leq k \leq 2^{t+m}$, $\{\bar{b}_{t;2^{t+m}(2i-1)+k}\}_i$ is a family of i.i.d. In this case, Proposition 5.3.1 gives a uniform constant K_3' such that

$$\mathbb{E}[b^3_{t;2^{t+m}+k}] \leq K_3'.$$

By taking $K_3 = 8K_3'$, we also have

$$\mathbb{E}\,|\bar{b}_{t;2^{t+m}+k}|^3 \leq \mathbb{E}(|b_{t;2^{t+m}+k}| + |\,\mathbb{E}\, b_{t;2^{t+m}+k}|)^3 \leq K_3.$$

Let us now define

$$E_{t,k,i} := \left\{\omega : |\bar{b}_{t;2^{t+m}(2i-1)+k}| > \frac{K'\sqrt{2^{t+m}(2i-1)}}{2^{t/8}}\right\}.$$

Then for $Y_{t,k} = |\bar{b}_{t;2^{t+m}+k}|/(2^{3t/8+m/2}K')$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} \mathbb{P}[E_{k,t,i}] &\leq \sum_{i=1}^{\infty} i \cdot \mathbb{P}\left\{\frac{K'\sqrt{2^{t+m}i}}{2^{t/8}} < |\bar{b}_{t;2^{t+m}+k}| \leq \frac{K'\sqrt{2^{t+m}(i+1)}}{2^{t/8}}\right\} \\
&\leq \int Y_{t,k}^2 1_{Y_{t,k}\geq 1}\, d\,\mathbb{P} \leq \int Y_{t,k}^3\, d\,\mathbb{P} \leq \frac{1}{2^{9t/8+3m/2}K'^3}\,\mathbb{E}\,|\bar{b}_{t;2^{t+m}+k}|^3 \\
&\leq \frac{K_3}{K'^3} 2^{-9t/8-3m/2}.
\end{aligned}$$

We sum them up to deduce

$$\sum_{t=1}^{\infty} \sum_{0\leq k\leq 2^t} \sum_{i=1}^{\infty} \mathbb{P}[E_{k,t,i}] < \infty.$$

Then by Borel-Cantelli, we conclude that for almost every ω,

$$|\bar{b}_{t;n}(\omega)| \leq \frac{K'\alpha(n)}{2^{t/8}}$$

for all t for all but finitely many n. Hence, for those ω we have

$$\frac{1}{\alpha(n)}\left|\sum_t \bar{b}_{t;n}\right| \leq 16K'$$

eventually. $\square$

We now finish the proof of the LIL. Fix $K' > 0$ and let $T > 0$ be as in Claim 6.4.1. The classical LIL tells us that

$$\limsup_n \frac{1}{\alpha(n)}\left|\sum_{i=1}^{\lfloor n/2^{m+t+1}\rfloor} \bar{b}_{t,2i-1}\right| \leq \frac{4K}{\sqrt{2^{m+t+1}}} \quad \text{a.s.}$$

for each $t \leq T$. Combining this with Claim 6.4.1 and Claim 6.4.2 gives

$$\limsup_n \frac{1}{\alpha(n)}\left|\sum_{2^{m+t}\leq n}\left[\bar{b}_{t,2\lfloor n/2^{m+t+1}\rfloor+1;n} + \sum_{i=1}^{\lfloor n/2^{m+t+1}\rfloor} \bar{b}_{t,2i-1}\right]\right| \leq K' + \frac{20K}{\sqrt{2^m}}$$

outside a set with probability at most K'. This is promoted to the almost sure statement by sending $K' \to 0$. Finally, the classical LIL implies that

$$\limsup_n \pm\frac{1}{\alpha(n)}\sum_{i=1}^{\lfloor n/2^m\rfloor} \bar{Y}_{0,i} = \sigma_m \quad \text{a.s.}$$

Together with the fact $\frac{1}{\alpha(n)}\bar{Y}_{0,\lfloor n/2^m\rfloor+1;n} \to 0$ a.s., we conclude that

$$\limsup_n \pm\frac{1}{\alpha(n)}[d(o,\omega_n\, o) - \mathbb{E}[d(o,\omega_n\, o)]] \in \left[\sigma_m - \frac{20K}{\sqrt{2^m}}, \sigma_m + \frac{20K}{\sqrt{2^m}}\right] \quad \text{a.s.}$$

Since $\sigma_m \to \sigma$ as $m \to \infty$, the desired conclusion follows.

6.5 Geodesic tracking

Given a random path $\omega = (\omega_n)_n$ with the set of eventual pivotal times $\mathcal{Q}(\omega) = \{i(1) < i(2) < \ldots\}$, we consider the concatenation $\Gamma = \Gamma(\omega)$ of

$$(\eta_1, \eta_2, \ldots) := ([o, \omega_{i(1)}\, o], [\omega_{i(1)}\, o, \omega_{i(1)+M_0}\, o], [\omega_{i(1)+M_0}\, o, \omega_{i(2)}\, o], [\omega_{i(2)}\, o, \omega_{i(2)+M_0}\, o], \ldots).$$

By Lemma 3.1.7, Γ is a quasigeodesic. We now show the geodesic tracking with doubled exponent.

Proposition 6.5.1. *Suppose that μ has finite p-th moment for some $p > 0$. Then for almost every sample path $\omega = (\omega_n)_n$, we have*

$$\lim_{k \to \infty} \frac{d(\omega_k\, o, \Gamma)}{k^{1/2p}} = 0.$$

Proof. By Corollary 5.3.2, $\min[d(o, \omega_\upsilon\, o), d(o, \check{\omega}_{\check{\upsilon}} o)]^{2p}$ is dominated by an integrable RV. This implies that

$$\sum_k \mathbb{P}\left(\min[d(o, \omega_\upsilon\, o), d(o, \check{\omega}_{\check{\upsilon}} o)] > g(k)\right) < \infty \tag{6.5.1}$$

for some g such that $\lim_k g(k)/k^{1/2p} = 0$. Note that the probabilities in the summation do not change after the Bernoulli shift T. Note also that $\mathbb{P}(\max\{\upsilon, \check{\upsilon}\} \geq k)$ is summable and is invariant under the Bernoulli shift. By the Borel-Cantelli lemma, we deduce the following for a.e. $(\check{\omega}, \omega)$. For each large k, there exists $j = j(k) \in \mathbb{Z}$ such that $|j| \leq k$, $d(\omega_k\, o, \omega_{k+j}\, o) \leq g(k)$ and either:

1. there exists $0 < i \leq j - M_0$ such that
   - $\alpha := (g_{k+i+1}, \ldots, g_{k+i+M_0})$ is a Schottky sequence,
   - $(\omega_k\, o, \omega_{k+i}\, \Gamma(\alpha), \omega_{k+n}\, o)$ is D_1-aligned for all $n \geq j$, and
   - $(\omega_{k-n'}\, o, \omega_{k+i}\, \Gamma(\alpha))$ is D_2-aligned for all $n' \geq 0$,

   or;
2. there exists $0 > i \geq j + M_0$ such that
   - $\alpha := (g_{k+i}^{-1}, g_{k+i-1}^{-1}, \ldots, g_{k+i-M_0+1}^{-1})$ is a Schottky sequence,
   - $(\omega_k\, o, \omega_{k+i}\, \Gamma(\alpha), \omega_{k+n}\, o)$ is D_1-aligned for all $n \leq j$,
   - $(\omega_{n'+k}\, o, \omega_{k+i}\, \Gamma(\alpha))$ is D_2-aligned for all $n' \geq 0$.

The first case is where j equals $\upsilon(T^k(\check{\omega}, \omega))$ and the second case is where j equals $-\check{\upsilon}(T^k(\check{\omega}, \omega))$. In both cases, the second item for $n = j$ leads to

$$d(\omega_k\, o, \omega_{k+i}\, \Gamma(\alpha)) \leq d(\omega_k\, o, \omega_{k+j}\, o) \leq g(k).$$

We now let $N = k + |j|$; note $i(N) > N$. In the first case of the dichotomy, $(o, \omega_{k+i}\, \Gamma(\alpha), \omega_{i(N)}\, o)$ is D_2-aligned. In the second case, $(\omega_{i(N)}\, o, \omega_{k+i}\, \Gamma(\alpha), o)$ is D_2-aligned. We now claim that $d(\eta_m, \omega_{k+i}\, \Gamma(\alpha))$ is bounded for some m.

The projections of the beginning point of η_1 and the terminating point of η_{2N-1} onto $\omega_{k+i}\, \Gamma(\alpha)$ are far away. Hence, one of the following holds.

(a) some η_m has a large projection on $\omega_{k+i}\, \Gamma(\alpha)$: more precisely, there exists η_m with endpoints $\{x_m, y_m\}$ such that

$$d(\pi_{\omega_{k+i}\, \Gamma(\alpha)}(x_m), \omega_{k+i}\, o) \leq 2K_0 + K_3 + 2E_0 + D_2,$$
$$d(\pi_{\omega_{k+i}\, \Gamma(\alpha)}(y_m), \omega_{k+i+M_0}\, o) \leq 2K_0 + K_3 + 2E_0 + D_2,$$

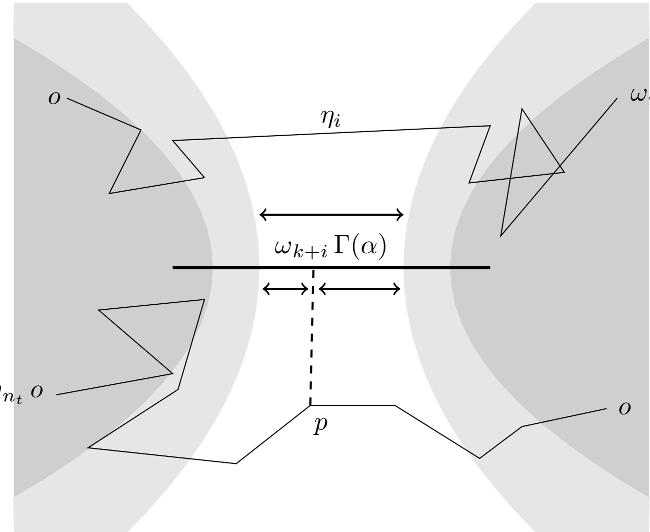

Figure 6.2: Dichotomy in the proof of Proposition 6.5.1. o and $\omega_{n_t}\, o$ are distant when seen from $\omega_{k+i}\,\Gamma(\alpha)$, so either an η_i is seen large (the upper case) or an endpoint p of some η_i is seen in the middle (the lower case).

(b) an endpoint p of some η_m projects onto $\omega_{k+i}\,\Gamma(\alpha)$ in the middle, i.e.,

$$d(\pi_{\omega_{k+i}\,\Gamma(\alpha)}(p), \omega_{k+i}\, o), d(\pi_{\omega_{k+i}\,\Gamma(\alpha)}(p), \omega_{k+i+M_0}\, o) > 2K_0 + K_3 + 2E_0 + D_2.$$

Recall that

$$d(\omega_{k+i}\, o, \omega_{k+i+M_0}\, o) \geq 2\left(\frac{M_0}{K_0} - K_0\right) \geq 6K_0 + 2K_3 + 4E_0 + 2D_2.$$

Hence, in Case (a), we deduce $d\left(\pi_{\omega_{k+i}\,\Gamma(\alpha)}(x_i), \pi_{\omega_{k+i}\,\Gamma(\alpha)}(y_i)\right) \geq 2K_0$ and η_i is within a neighborhood of $\omega_{k+i}\,\Gamma(\alpha)$ by the K_0-BGIP of $\Gamma(\alpha)$.

In Case (b), recall that the Schottky axes at eventual pivotal times are parts of a D_0-aligned sequence; by Proposition 3.1.5, p is within distance E_0 from some $q \in [o, \omega_{n_t}\, o]$. Then q also projects onto $\omega_{k+i}\,\Gamma(\alpha)$ in the middle:

$$d(\pi_{\omega_{k+i}\,\Gamma(\alpha)}(q), \omega_{k+i}\, o), d(\pi_{\omega_{k+i}\,\Gamma(\alpha)}(q), \omega_{k+i+M_0}\, o) > 2K_0 + D_2.$$

Since the projections of $[o, q]$ and $[q, \omega_{n_t}\, o]$ onto $\omega_{k+i}\,\Gamma(\alpha)$ are both large, we can apply Lemma 2.2.5 and obtain $q_1 \in [o, q]$, $q_2 \in [q, \omega_{n_t}\, o]$ such that $d(q_1, \pi_{\omega_{k+i}\,\Gamma(\alpha)}(q)), d(q_2, \pi_{\omega_{k+i}\,\Gamma(\alpha)}(q)) < K_3$. This forces that p is also near $\omega_{k+i}\,\Gamma(\alpha)$. $\square$

In the previous lemma, we only assumed $p > 0$. Namely, sublinear tracking occurs even when μ has finite $(1/2)$-th moment only. When μ has finite exponential moment, the exact same proof works with $g(k) = C \log k$ for some suitable C. This leads to the following:

Proposition 6.5.2. *Suppose that μ has finite exponential moment. Then there exists $C > 0$ such that for almost every sample path $\omega = (\omega_n)_n$, we have*

$$\limsup_{k \to \infty} \frac{d(\omega_k\, o, \Gamma)}{\log k} \leq C.$$

Chapter 7. Translation length

In this chapter, we develop the theory for translation length. We present two approaches: the first one utilizing the deviation inequalities, and the second one with more concrete access to the pivotal times.

7.1 First approach

In Chapter 5, we defined RVs υ, $\check{\upsilon}$ and the probabilistic estimations on them. Using them, we prove the following theorem:

Theorem G. *Let ω be the random walk generated by a non-elementary measure μ on G.*

1. *If μ has finite p-th moment for some $p > 0$, then*

$$\lim_{n\to\infty} \frac{1}{n^{1/2p}}[d(o, \omega_n\, o) - \tau(\omega_n)] = 0 \quad a.s.$$

2. *If μ has finite first moment, then there exists $K > 0$ such that*

$$\limsup_{n\to\infty} \frac{1}{\log n}[d(o, \omega_n\, o) - \tau(\omega_n)] \leq K \quad a.s.$$

Proof. Let $\kappa_1, K_1 > 0$ be as in Lemma 5.2.3.

Suppose first that μ has finite p-th moment for some $p > 0$. Let Z be an integrable RV that dominates $\min\{d(o, \omega_\upsilon\, o), d(o, \check{\omega}_{\check{\upsilon}} o)\}^{2p}$. Let also $\kappa_1, K_1 > 0$ be the constants as in Lemma 5.2.3.

Let us fix $n > 0$. We temporarily define

$$\begin{aligned} h_{nk+i} &:= g_i \quad (k \in \mathbb{Z}, i \in \{1, \ldots, n\}), \\ \omega_i &:= \begin{cases} h_1 \cdots h_i & i \geq 0, \\ h_0^{-1} \cdots h_{i+1}^{-1} & i < 0. \end{cases} \end{aligned}$$

For $t = 0, 1, 2, 3$, we also define

$$\begin{matrix} g_{i;t} := h_{i+\lfloor nt/4 \rfloor} & \check{g}_{i;t} := h^{-1}_{\lfloor nt/4 \rfloor - i+1} \\ \omega_{i;t} := g_{1;t} \cdots g_{i;t}, & \check{\omega}_{i;t} := \check{g}_{1;t} \cdots \check{g}_{i;t}. \end{matrix} \qquad (i = 1, \ldots, \lfloor n/2 \rfloor)$$

$(\check{g}_{i;t}, g_{i;t})_i$'s for $t = 0, 1, 2, 3$ have the same distribution with $(\check{g}_i, g_i)_i$, although they are not mutually independent. Let

$$\upsilon_{(t)} := \upsilon\left((\check{\omega}_{i;t})_{0 \leq i \leq \lfloor n/2 \rfloor}, (\omega_{i;t})_{0 \leq i \leq \lfloor n/2 \rfloor}\right), \quad \check{\upsilon}_{(t)} := \check{\upsilon}\left((\check{\omega}_{i;t})_{0 \leq i \leq \lfloor n/2 \rfloor}, (\omega_{i;t})_{0 \leq i \leq \lfloor n/2 \rfloor}\right)$$

and observe that

$$\begin{aligned} \mathbb{P}\left(A_{n;t} := \{\omega : \max\{\upsilon_{(t)}, \check{\upsilon}_{(t)} \geq n/10\}\right) &\leq K_1 e^{-\kappa_1 n/10}, \\ \min\left\{d\left(o, \omega_{\upsilon_{(0)}}\, o\right), d\left(o, \check{\omega}_{\check{\upsilon}_{(0)}} o\right)\right\}^{2p} &\leq Z. \end{aligned}$$

We now claim that for $\omega \notin A_n^{(0)} \cup A_n^{(1)} \cup A_n^{(2)} \cup A_n^{(3)}$, we have

$$[d(o, \omega_n\, o) - \tau(\omega_n)]^{2p} \leq 2^{2p} Z.$$

We explain the case that $d\left(o, \omega_{\upsilon_{(0)}}\, o\right)^{2p} \le Z$, since the other case can be discussed in a similar manner.

By the definition of $\upsilon_{(t)}$, there exist $i(0), i(1), i(2), i(3)$ such that $nt/4 \le i(t) \le nt/4 + \upsilon_{(t)} - M_0$ and the following holds. If we define

$$s_t = (g_{i(t)+1}, \ldots, g_{i(t)+M_0}),$$

then s_t's are Schottky sequences and

$$\left(\omega_{\lfloor nt/4 \rfloor - j}\, o,\ \omega_{i(t)}\, \Gamma(s_t),\ \omega_{\lfloor nt/4 \rfloor + k}\right)$$

is D_2-aligned for $0 \le j \le n/2$ and $\upsilon_{(t)} \le k \le n/2$. Note also that $\upsilon_{(t)} \le n/10$ since w does not belong to any of $A_{n;t}$. This implies that

$$\left(o,\ \omega_{i(0)}\, \Gamma(s_0),\ \ldots,\ \omega_{i(3)}\, \Gamma(s_3),\ \omega_n\, \omega_{i(0)}\, \Gamma(s_1),\ \ldots,\ \omega_n^{k-1}\, \omega_{i(3)}\, \Gamma(s_3),\ \omega_n^k\, o\right)$$

is D_2-aligned for each $k > 0$. Using Proposition 3.1.5 we can control the Gromov products among points, which imply

$$d(o, \omega_n^k\, o) \ge d(o, \omega_{i(0)}\, o) + \sum_{j=1}^{k-1} d\left(\omega_n^{j-1}\, \omega_{i(0)}\, o, \omega_n^j\, \omega_{i(0)}\, o\right) + d(o, \omega_n^{k-1}\, \omega_{i(0)}\, o, \omega_n^k\, o) - (k+1)E_0.$$

Hence, we have

$$\begin{aligned} \tau(\omega_n) &\ge d(\omega_{i(0)}\, o, \omega_n\, \omega_{i(0)}\, o) - E_0, \\ [d(o, \omega_n\, o) - \tau(\omega_n)]^{2p} &\le (2d(o, \omega_i\, o) + E_0)^{2p}. \end{aligned}$$

Note that $(o, \omega_{i(0)}\, \Gamma(s_0), \omega_{\upsilon_{(0)}}\, o)$ is also D_2-aligned so we have

$$\begin{aligned} d(o, \omega_{i(0)}\, o) &\le d(o, \omega_{\upsilon_{(0)}}\, o) - 10E_0, \\ [d(o, \omega_n\, o) - \tau(\omega_n)]^{2p} &\le (2d(o, \omega_i\, o))^{2p} \le 2^{2p} d(o, \omega_{\upsilon_{(0)}}\, o)^{2p} \le 2^{2p} Z. \end{aligned}$$

This implies

$$\begin{aligned} \mathbb{P}(d(o, \omega_n\, o) - \tau(\omega_n) \ge Cn^{1/2p}) &= \mathbb{P}\left([d(o, \omega_n\, o) - \tau(\omega_n)]^{2p} \ge C^{2p} n\right) \\ &\le \mathbb{P}\left(2^{2p} Z \ge C^{2p} n\right) + 2Ke^{-\kappa n/10}. \end{aligned}$$

Since Z is integrable, the above probability is summable and the Borel-Cantelli lemma leads to the conclusion.

Now suppose that μ has finite first moment. This time, we define

$$A_{n;t} := \{\omega : \upsilon_{(t)} \ge K' \log n\}$$

for some large K' such that $\sum_n K_1 e^{-\kappa_1 K' \log n} < +\infty$. Then the Borel-Cantelli lemma tells us that almost every path ω eventually lies outside $A_n^{(1)} \cup A_n^{(2)} \cup A_n^{(3)} \cup A_n^{(4)}$, say for $n \ge N$. In such case, we have

$$d(o, \omega_n\, o) - \tau(\omega_n) \le d(o, \omega_{\upsilon_{(0)}}\, o) \le d(o, \omega_{K' \log n}\, o)$$

for $n \ge N$. Finally, the subadditive ergodic theorem tells us that $d(o, \omega_m\, o) \le 2\lambda m$ eventually holds for almost every path. Hence we conclude that

$$d(o, \omega_n\, o) - \tau(\omega_n) \le 2\lambda K' \log n$$

eventually for almost every path. $\square$

Corollary 7.1.1 (SLLN for finite first moment)**.** *Let ω be the random walk generated by a non-elementary measure μ on G with finite first moment. Then*

$$\lim_{n} \frac{1}{n}\tau(\omega_n) = \lambda \tag{7.1.1}$$

for almost every ω, where $\lambda = \lambda(\mu)$ is the escape rate of μ.

Corollary 7.1.2 (CLT)**.** *Let ω be the random walk generated by a non-elementary measure μ on G. If μ has finite second moment, then $\frac{1}{\sqrt{n}}(\tau(o, \omega_n\, o) - n\lambda)$ and $\frac{1}{\sqrt{n}}(d(o, \omega_n\, o) - n\lambda)$ converge to the same Gaussian distribution $\mathscr{N}(0, \sigma(\mu)^2)$ in law. We also have*

$$\limsup_{n\to\infty} \pm \frac{\tau(o, \omega_n\, o) - \lambda n}{\sqrt{2n \log\log n}} = \sigma(\mu) \quad \textit{almost surely.}$$

Theorem G also implies Corollary 7.1.1 for measures with finite (1/2)-th moment, and the converse of CLT for measures with finite (1/4)-th moment. However, for general non-elementary measures, the SLLN and the converse of CLT cannot be deduced from Theorem G and we need more explicit information. We now present the second approach that explicitly refers to the pivotal time structure.

7.2 Second approach

We discuss pivoting on random paths for translation length. Given $(w_j)_{j=0}^{\infty}$, $(v_j)_{j=0}^{\infty}$, we consider an equivalence class $\mathcal{E} \subseteq S^{4n}$ made by pivoting. $\mathcal{E}$ has a well-defined set of pivotal times $P_n(\mathcal{E}) = \{i(1), \ldots, i(M)\}$, and a choice $s \in \mathcal{E}$ is determined by the choices $(\alpha_{i(l)}, \beta_{i(l)}, \gamma_{i(l)})_{l=1}^{M}$. We also denote $w_{n+1,2}^{-}(s)$ by w for convenience throughout the subsection.

Recall that we have constructed $\tilde{S}_l \subseteq S^3$ that depends on $\left(\alpha_{i(j)}, \beta_{i(j)}, \gamma_{i(j)}\right)_{j=1}^{l-1}$. We now define new subsets:

$$\begin{aligned} S_1^*(s) &= S_1^*(\gamma_{i(M)}), \\ S_M^*(s) &= S_M^*(\alpha_{i(1)}, \gamma_{i(M)}), \\ S_2^*(s) &= S_2^*(\alpha_{i(1)}, \beta_{i(1)}, \gamma_{i(1)}, \alpha_{i(M)}, \beta_{i(M)}, \gamma_{i(M)}, \gamma_{i(M-1)}), \\ S_{M-1}^*(s) &= S_{M-1}^*(\alpha_{i(1)}, \beta_{i(1)}, \gamma_{i(1)}, \alpha_{i(M)}, \beta_{i(M)}, \gamma_{i(M)}, \alpha_{i(2)}, \gamma_{i(M-1)}), \\ &\vdots \end{aligned}$$

for $1 \le k \le \lfloor M/2 \rfloor$. To define them we first consider

$$\begin{aligned} \phi_k &:= (w_{i(M-k+1),0}^{-})^{-1} w w_{i(k),2}^{-} \\ &= v_{i(M-k+1)} c_{i(M-k+1)} d_{i(M-k+1)} w_{i(M-k+1)} \cdots a_n b_n v_n c_n d_n w_n \\ &\quad \cdot w_0 a_1 b_1 v_1 c_1 d_1 w_1 \cdots a_{i(k)-1} b_{i(k)-1} v_{i(k)-1} c_{i(k)-1} d_{i(k)-1} w_{i(k)-1} \end{aligned}$$

for $1 \le k \le \lfloor M/2 \rfloor$. It is clear that ϕ_k depends on $\gamma_{i(M-k+1)}$, $\alpha_{i(M-k+2)}$, $\ldots$, $\gamma_{i(M)}$, $\alpha_{i(1)}$, $\beta_{i(1)}$, $\ldots$, $\gamma_{i(k-1)}$. Then we set

$$\begin{aligned} S_k^*(s) &:= \left\{ \alpha_{i(k)} \in S \quad : \left(w^{-1} y_{i(M-k+1),0}^{-},\ \Upsilon(\alpha_{i(k)}) \right) \text{ is } K_0\text{-aligned} \right\}, \\ S_{M-k+1}^*(s) &:= \left\{ \beta_{i(M-k+1)} \in S : \left(w^{-1}\, \Upsilon(\beta_{i(M-k+1)}), y_{i(k),1}^{-} \right) \text{ is } K_0\text{-aligned} \right\}. \end{aligned}$$

Here, the conditions above can be expressed as

$$\operatorname{diam}\left(\pi_{\Gamma(\alpha_{i(k)})}(\phi_k^{-1} o) \cup o \right) < K_0, \tag{7.2.1}$$

$$\operatorname{diam}\left(\pi_{\Gamma^{-1}(\beta_{i(M-k+1)})}(\phi_k a_{i(k)} o) \cup o \right) < K_0, \tag{7.2.2}$$

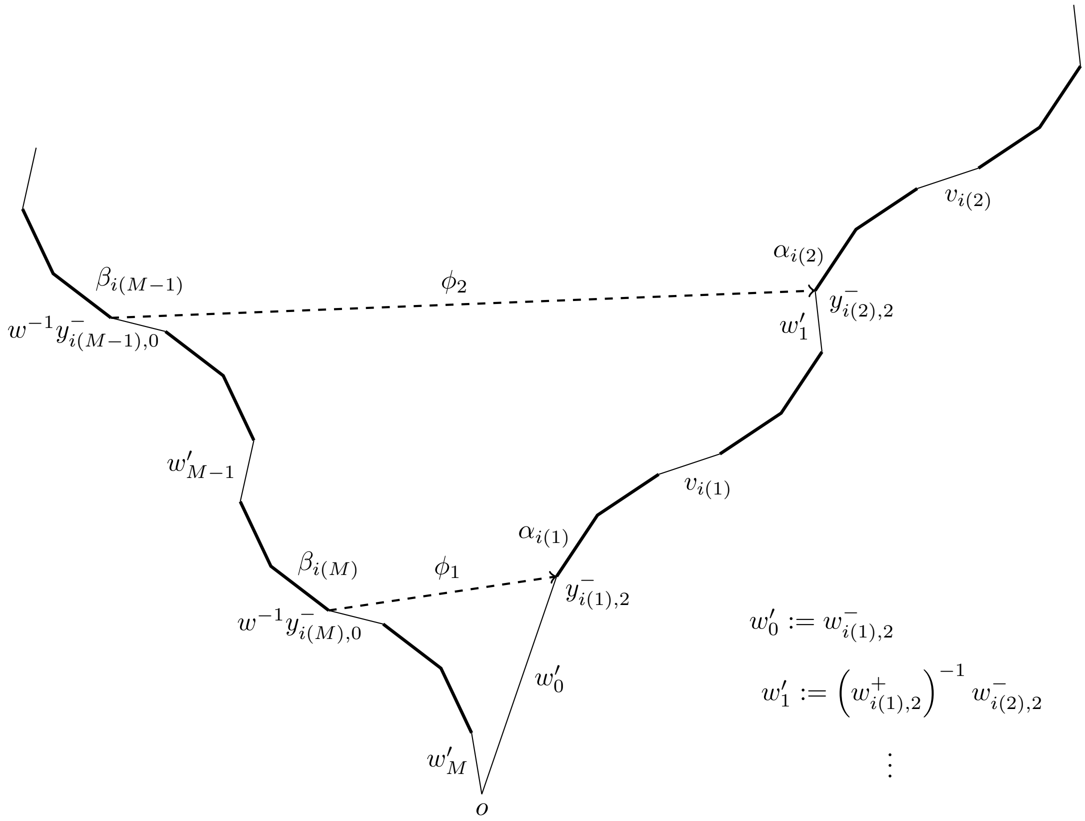

Figure 7.1: Defining ϕ_k's used in the pivoting for translation length.

respectively. For each l, $S \setminus S_l^*(s)$ consists of at most 1 element thanks to the property of the Schottky set S.

Lemma 7.2.1. *Let $1 \le k \le M/2$. Suppose that $s = (\alpha_{i(l)}, \beta_{i(l)}, \gamma_{i(l)})_{l=1}^M \in \mathcal{E}_n$ satisfies*

$$\alpha_{i(k)} \in S_k^*(s), \quad \beta_{i(M-k+1)} \in S_{M-k+1}^*(s).$$

Then $w = w^-_{n+1,2}$ is a BGIP isometry and satisfies

$$\tau(w) \ge d(o, wo) - \left[d\big(o, y^-_{i(k),1}\big) + d\big(y^-_{i(M-k+1),1}, wo\big)\right] - 4E_0.$$

Proof. Suppose that $s \in \mathcal{E}_n$ satisfies the hypothesis. Then by Lemma 3.1.2, $\big(w^{-1}\,\Upsilon(\beta_{i(M-k+1)}, \Upsilon(\alpha_{i(k)})\big)$ is D_0-aligned. Recall also that

$$\big(\Upsilon(\alpha_{i(k)}), \Upsilon(\beta_{i(k)}), \Upsilon(\gamma_{i(k)}), \Upsilon(\delta_{i(k)}), \ldots, \Upsilon(\alpha_{i(M-k+1)}), \Upsilon(\beta_{i(M-k+1)}), \Upsilon(\gamma_{i(M-k+1)}), \Upsilon(\delta_{i(M-k+1)})\big)$$

is a subsequence of a D_1-aligned sequence by Lemma 4.1.1. Hence, if we define

$$\kappa_{2t+1} := w^t\,\Upsilon(\alpha_{i(k)}),$$
$$\kappa_{2t+2} := w^t\,\Upsilon(\beta_{i(M-k+1)})$$

for $t \in \mathbb{Z}$, we observe that $(o, \kappa_1, \kappa_2, \ldots, \kappa_{2i-1}, \omega^i\, o)$ is a subsequence of a D_1-aligned sequence. Proposition 3.1.5 then tells us that the Gromov products among the endpoints of κ_i's are bounded by E_0.

Hence, we have

$$d(o, w^i o) \geq d(o, y^-_{i(k),1}) + \sum_{j=1}^{i} d(w^{j-1} y^-_{i(k),1}, w^{j-1} y^-_{i(M-k+1),1}) + \\ \sum_{j=1}^{i-1} d(w^{j-1} y^-_{i(M-k+1),1}, w^j y^-_{i(k),1}) + d(w^{i-1} y^-_{i(M-k+1),1}, w^i o) - 4iE_0.$$

Dividing the both hand sides by i, we conclude that

$$\begin{aligned} \tau(w) &\geq d(y^-_{i(k),1}, y^-_{i(M-k+1),1}) - 4E_0 \\ &\geq d(o, wo) - d(o, y^-_{i(k),1}) - d(y^-_{i(M-k+1),1}, o) - 4E_0. \end{aligned} \tag{7.2.3}$$

Moreover, since $[y^-_{i(k),1}, y^-_{i(M-k+1),1}]$ is E_0-witnessed by Schottky axes and longer than $4E_0$, Inequality 7.2.3 also tells us that $\tau(w) > 0$. Similarly we have $\tau(w^{-1}) > 0$, so w is a bi-quasigeodesic.

It remains to show that the orbit of w has BGIP. Since $\{w^i o\}_i$ and $(\kappa_i)_i$ are close to each other, it suffices to establish the BGIP of the latter one. Since $(\kappa_i)_i$ is D_1-aligned sequence of Schottky axes, we can apply Lemma 3.1.6. In particular, consider $x, y \in X$ and $n \in \mathbb{Z}$. If $J_0(x; (\kappa_i)_i, D_2)$ contains an element smaller than (larger than, resp.) n and $J_0(y; (\kappa_i)_i, D_2)$ contains an element larger than (smaller than, resp.) n, then the projection of $[x, y]$ onto κ_n is large and $[x, y]$ passes near κ_n. Such a phenomenon happens when the diameter of the projection of $[x, y]$ onto $\cup_i \kappa_i$ exceeds $\max\{\mathrm{diam}(\kappa_{i-1} \cup \kappa_i) : i \in \mathbb{Z}\}$, which is actually the maximum among finitely many numbers and thus finite. Hence, $\cup_i \kappa_i$ has BGIP and we are led to the conclusion. $\square$

We now estimate the probability for the event described in Lemma 7.2.1. Given a choice

$$\bar{s} = (\bar{\alpha}_{i(l)}, \bar{\beta}_{i(l)}, \bar{\gamma}_{i(l)})_{l=1,\ldots,k-1,M-k+2\ldots,M} \in \tilde{S}_{i(1)} \times \cdots \times \tilde{S}_{i(k-1)} \times \tilde{S}_{i(M-k+2)} \times \cdots \times \tilde{S}_{i(M)},$$

we define

$$S_k^\dagger := \left\{ \begin{array}{c} (\alpha_{i(k)}, \beta_{i(k)}, \gamma_{i(k)}, \alpha_{i(M-k+1)}, \beta_{i(M-k+1)}, \gamma_{i(M-k+1)}) \in \tilde{S}_{i(k)} \times \tilde{S}_{i(M-k+1)} \\ : \quad \alpha_{i(k)} \in S_k^*(\bar{s}, \gamma_{M-k+1}) \text{ and} \\ \beta_{i(M-k+1)} \in S^*_{M-k+1}(\bar{s}, \alpha_{i(k)}, \gamma_{i(M-k+1)}) \end{array} \right\}$$

Then we have the following:

Lemma 7.2.2. *For each $1 \leq k \leq \lfloor M/2 \rfloor$, the cardinality of $\tilde{S}_k^\dagger$ is at least $(\#S)^6 - 8(\#S)^5$.*

Proof. First, there are at least $(\#S - 1)$ choices of $\gamma_{i(k)}$ and $(\#S - 1)$ choices of $\gamma_{i(M-k+1)}$ in S that satisfy Inequality 4.1.2. Fixing those choices, at least $(\#S - 1)$ choices of $\beta_{i(k)}$ in S satisfy Inequality 4.1.4. Finally, fixing those choices, there are at most 1 choice of $\alpha_{i(k)}$ in S that violates Inequality 4.1.5 and at most 1 choice that violates Inequality 7.2.1. In other words, at least $(\#S - 2)$ choices of $\alpha_{i(k)}$ satisfy both inequalities.

Fixing the above choices, at most 1 choices of $\beta_{i(M-k+1)}$ in S violates Inequality 4.1.4 and at most 1 choice in S violates Inequality 7.2.2. In other words, at least $(\#S - 2)$ choices of $\beta_{i(k)}$ satisfy both inequalities. Finally, fixing those choices, there are at least $(\#S - 1)$ choices of $\alpha_{i(k)}$ in S that satisfy Inequality 4.1.5. Overall, we conclude that $\tilde{S}^*_{i(k)}$ has cardinality at least $(\#S - 1)^4 (\#S - 2)^2 \geq (\#S)^6 - 8(\#S)^5$. $\square$

7.3 SLLN with and without moment conditions

In Section 4.3, we proved that random walks escapes to infinity almost surely. This leads to the SLLN for finite first moment. In this section, we obtain finer estimates with and without moment conditions, by realizing Gouëzel's estimation in [Gou21, Section 5] with Schottky sets.

Lemma 7.3.1 ([Gou21, Proposition 5.1]). *Let ρ_1, ρ_2, ... be probability measures on G and R be a nonnegative RV such that for all i and $M \geq 0$ we have*

$$\mathbb{P}_{\rho_i}(d(o, go) \geq M) \geq \mathbb{P}(R \geq M).$$

Now let $w_0, w_1, \ldots \in G$ and s_i be independent RVs that are sampled according to $\mu_S^{*2} * \rho_i * \mu_S^{*2}$. Let $w = w_0 s_1 w_1 \cdots s_n w_n$ and $y = wo$. Then there exists $K = K(N_0) > 0$ such that for all $M \geq 0$,

$$\mathbb{P}(w_n \text{ does not have BGIP or } \tau(w_n) \leq M) \leq \mathbb{P}(R_1 + \cdots + R_{(1-150/N_0)n} \leq M) + e^{-n/K}.$$

Proof. Following the notation in Chapter 4, we consider RVs $\{\alpha_i\}_{i=1}^n$, $\{\beta_i\}_{i=1}^n$, $\{\gamma_i\}_i$, $\{\delta_i\}_i$ with the law of μ_S, and v_i with the law of ρ_i for $i = 1, \ldots, n$, all independent. We also let a_i, b_i, c_i, d_i as in Equality 4.1.1. Then $a_i b_i v_i c_i d_i$ serves as s_i.

Let $\mathcal{E}^{(1)}$ be an equivalence class by the pivoting as in the second model of Section 4.2, with at least $(1-10/N_0)n$ pivoting times. Fix the values of v_i's, and then the values of $(\alpha_i, \beta_i, \gamma_i)$'s, at the first and the last n/N_0-pivotal times of $\mathcal{E}^{(1)}$. This process divides $\mathcal{E}^{(1)}$ into finer equivalence classes $\{\mathcal{E}_1^{(2)}, \ldots, \mathcal{E}_N^{(2)}\}$ made by pivoting at the intermediate $n' \geq (1 - 12/N_0)n$ pivotal times; we denote these intermediate pivotal times by $\{i(1) < i(2) < \ldots < i(n')\}$. Let us consider the condition for $\mathcal{E}_i^{(2)}$'s:

$$w_n \in \mathcal{E}_i^{(2)} \text{ are all BGIP with } \tau(\omega_n) \geq d(y^-_{i(1),0}, y^-_{i(n'),2}). \tag{7.3.1}$$

Then Lemma 7.2.2 asserts that the equivalence classes $\mathcal{E}_i^{(2)}$ that satisfy the above condition take up at least $1 - (8/N_0)^{n/N_0}$ of the probability of $\mathcal{E}^{(1)}$.

Let us now fix an equivalence class $\mathcal{E}^{(2)}$ that satisfy Condition 7.3.1. For $s \in \mathcal{E}^{(2)}$ we consider points

$$(x_{3k-2}, x_{3k-1}, x_{3k}) := (y^-_{i(k),0}, y^+_{i(k),0}, y^+_{i(k),1})$$

for $k = 1, \ldots, n'$. Then Proposition 3.1.4 and Lemma 4.1.1 asserts that $(x_i, x_k)_{x_j} \leq E_0$ for all $i < j < k$. This implies that

$$\begin{aligned}
\tau(\omega_n) &\geq \sum_{i=1}^{3n'} d(x_{i-1}, x_i) - 2E_0 \cdot 3n' \\
&\geq \sum_{k=1}^{n'} [d(x_{3k-2}, x_{3k-1}) + d(x_{3k-1}, x_{3k}) - 6E_0] \\
&\geq \sum_{k=1}^{n} \left[d(o, v_{i(k)}o) + (M_0/K_0 - K_0 - 6E_0)\right] \geq \sum_{k=1}^{n'} d(o, v_{i(k)}o).
\end{aligned}$$

We now estimate the expectation of $d(o, v_{i(k)}o)$ for each k. The proof of Lemma 4.1.4 implies that for each choice of $v_{i(k)}$, we have $(\alpha_{i(k)}, \beta_{i(k)}, v_i, \gamma_{i(k)}) \in \tilde{S}'_{i(k)}$ for all $(\alpha_{i(k)}, \beta_{i(k)}, \gamma_{i(k)}) \in S^3$ except at most $3(\#S)^2$ choices. Hence, we have

$$\mathbb{P}\left[v_{i(k)} = g \,\middle|\, (\alpha_{i(k)}, \beta_{i(k)}, v_{i(k)}, \gamma_{i(k)}) \in \tilde{S}'_{i(k)}\right] \geq \frac{\rho_i(g) \cdot [(\#S)^3 - 3(\#S)^2]}{\#S^3} \geq \rho_i(g) \cdot \left(1 - \frac{3}{N_0}\right).$$

Consequently, $d(o, v_{i(k)}o)$ conditioned on $\mathcal{E}^{(2)}$ dominates $B_k R_k$; here, $\{B_k\}_k$ are Bernoulli RVs that have value 1 with probability $1 - 3/N_0$ and value 0 with probability $3/N_0$, $\{R_k\}_k$ have the same law with R, and $\{B_k, R_k\}_k$ are independent. This implies that

$$\mathbb{P}\left[\tau(\omega_n) \leq M \,\middle|\, \mathcal{E}^{(2)}\right] \leq \mathbb{P}\left[\sum_{k=1}^{n'} B_k R_k \leq M\right].$$

Note that $\sum_{k=1}^{n'} B_k \geq (1-6/N_0)n'$ outside a set of probability e^{-n'/K_3} for some $K_3 > 0$ that depends on N_0. Conditioned on the event where $B_{k_1}, \ldots, B_{k_{(1-6/N_0)n'}} = 1$ for some $1 \leq k_1 < \ldots < k_{(1-6/N_0)n'} \leq n'$, we have

$$\begin{aligned} \mathbb{P}\left[d(o, y(s)) \leq M | \mathcal{E}^{(2)}\right] &\leq \mathbb{P}\left[R_{k_1} + \ldots + R_{k_{(1-6/N_0)n'}} \leq M\right] \\ &\leq \mathbb{P}\left[R_1 + \ldots + R_{(1-100/N_0)n} \leq M\right]. \end{aligned}$$

This implies that

$$\mathbb{P}\left[d(o, y(s)) \leq M \,\middle|\, \mathcal{E}^{(2)}\right] \leq \mathbb{P}\left[R_1 + \ldots + R_{(1-100/N_0)n} \leq M\right] + e^{-n'/K_3}.$$

Note that such $\mathcal{E}^{(2)}$ takes up large portion of $\mathcal{E}^{(1)}$, and we have

$$\mathbb{P}\left[d(o, y(s)) \leq M \,\middle|\, \mathcal{E}^{(1)}\right] \leq \mathbb{P}\left[R_1 + \ldots + R_{(1-100/N_0)n} \leq M\right] + e^{-n'/K_3} + (8/N_0)^{n/N_0}.$$

Finally, such $\mathcal{E}^{(1)}$ takes up large portion of the entire space and we have

$$\mathbb{P}\left[d(o, y(s)) \leq M \,\middle|\, \mathcal{E}^{(2)}\right] \leq \mathbb{P}\left[R_1 + \ldots + R_{(1-100/N_0)n} \leq M\right] + e^{-n'/K_3} + (8/N_0)^{n/N_0} + e^{-K_4 n}$$

where $K_4 = K$ is as in Corollary 4.1.8. □

Theorem 7.3.2. *Let ω be a non-elementary random walk with infinite first moment. Then for any $K > 0$, there exists $K' > 0$ such that*

$$\mathbb{P}(\tau(\omega_n) \leq Kn \text{ or } \omega_n \text{ does not have BGIP}) \leq K' e^{-n/K'}.$$

Proof. We employ the model with the decomposition

$$\mu^{(4M_0+1)} = \alpha(\mu_S^2 \times \mu \times \mu_S^2) + (1-\alpha)\nu.$$

and consider the independent RVs $\{\rho_i, \eta_i, \nu_i\}_i$. We have $\mathbb{P}(\mathcal{N}(\omega_n) \leq K_3 n) \leq K_3 e^{-n/K_3}$ for some $K_3 > 0$. Fixing the values of ρ_i's that make $\mathcal{N}(\omega) \geq K_3 n$, and also the values of ν_i's, we can now employ Lemma 7.3.1.

Lemma 7.3.1 implies that $\mathbb{P}(\tau(\omega_n) \leq Kn) \leq \mathbb{P}(R_1 + \ldots + R_{(1-150/N_0)K_3 n} \leq Kn)$, where R_i are independent copies of the RV $d(o, go)$ for g following the law of μ. Since we assumed that μ has infinite first moment, we know that the latter probability decays exponentially. To be explicit, one can truncate R_i at M so that $\min(R_i, M)$ has expectation greater than $\frac{K}{K_1(1-100/N_0)}$, and apply the large deviation theory for bounded variables. Hence, we conclude that

$$\mathbb{P}(\tau(\omega_n) \leq Kn | \mathcal{N}(\omega_n) \leq K_3 n)$$

decays exponentially. We now sum up the above conditional probability for various equivalence classes corresponding to the event $\{\mathcal{N}(\omega_n) \leq K_3 n\}$. Since $\mathbb{P}(\mathcal{N}(\omega_n) \leq K_3 n)$ also decays exponentially, we can finish the proof. □

Let us now establish an exponential bound for random walks with finite first moment.

Lemma 7.3.3 ([Gou21, Lemma 4.14]). *For each $\epsilon > 0$, there exists $C > 0$ such that*

$$\mathbb{P}\left(\inf_n d(x, \omega_n o) \geq d(x, o) - C\right) \geq 1 - \epsilon$$

holds for any $x \in X$.

Proof. Note that the pivotal time construction works even if we replace the beginning point o with arbitrary point $x \in X$. Indeed, it amounts to setting $z_0 = x$ instead of $z_0 = o$, and resetting z_n as x instead of o when there is no sequence qualifying Criterion (B) during the pivot construction. After this modification, we obtain $K > 0$ (independent of the choice of x) such that the following holds: for $\omega \in \Omega$ outside a set of probability $Ke^{-n/K}$, we have $d(x, \omega_m o) \geq d(x, \omega_{i_1} o)$ for all $m \geq n$ for some $i_1(\omega) < Kn$ (we may take i_1 to be the first element in $\mathcal{Q}_n(\omega)$). Given $\epsilon > 0$, we can now take n such that $Ke^{-n/K} < \epsilon/2$, and $C > 0$ such that $\max[d(o, \omega_i o) : i = 1, \ldots, n] \leq C$ outside a set of probability $\epsilon/2$. □

Theorem 7.3.4. *Let ω be a non-elementary random walk with finite first moment, and λ be its escape rate. Then for any $0 < K < \lambda$, there exists $K' > 0$ such that*

$$\mathbb{P}(\tau(\omega_n) \leq Kn \text{ or } \omega_n \text{ does not have BGIP}) \leq K' e^{-n/K'}.$$

Proof. Recall that the third model involves constants $0 < \alpha < 1$ and $K_{sleep} > 0$. We can also begin by taking large enough N_0 when deciding the Schottky set S. However, recall that $K_0 = K_0(N_0)$, D_0, D_1, L_1, E_1 and L_2 are all depending on N_0. Since we are requiring Inequality 3.2.5, this will also increase M_0. However, it will be apparent that the increase of M_0 does not harm the forthcoming argument. We now explain how small α should be and how large N_0, K_{sleep} should be.

Since $K < \lambda$, the subadditive ergodic theorem provides $K, \epsilon_1 > 0$ such that

$$\lambda' := \frac{1}{2M_0\, K_{sleep}} \mathbb{E}_{\mu^{*2M_0\, K_{sleep}}}[d(o, go)] > K + \epsilon_1 \tag{7.3.2}$$

holds when $2M_0\, K_{sleep} \geq K_4$. Moreover, we have

$$\frac{K}{1 - 200/N_0} < (1 - 2\epsilon_2)(K + \epsilon_1)$$

for large enough N_0 and small enough $\epsilon_2 > 0$. We first decide such large N_0 and an associated K_0-Schottky set S with $\#S \geq N_0$. We then take large enough $M_0 > 1$ that satisfies Inequality 3.2.5. At the moment, we can determine the decomposition as in Equation 4.4.1 for some $0 < \alpha < 1$ and non-elementary ν. Let $C = C(\epsilon_2)$ be as in Lemma 7.3.3 for this ν, and we take $K_5 > 0$ such that

$$\frac{K}{1 - 200/N_0} < (1 - 2\epsilon_2)(K + \epsilon_1) - \frac{C}{K_5}. \tag{7.3.3}$$

Finally, let $\{X_j\}_j$ be independent geometric RVs whose distributions satisfy

$$\mathbb{P}(X_j = i) = \alpha(1 - \alpha)^{i-1} \quad (i = 1, 2, \ldots).$$

Since X_j's have uniform exponential moment, There exists $K_6, K_7 > 0$ such that

$$\mathbb{P}\left(\sum_{j=1}^{m} X_j \geq K_6 m\right) \leq e^{-K_7 m}.$$

We then require K_{sleep} to be larger than K_4, K_5 and $N_0(K_6+1)$.

Given n, let $n' = \lfloor n/2M_0 \rfloor$ and

$$\begin{aligned}
w_{i-1} &:= g_{2M_0[t'_{i-1}+1]+1} \cdots g_{2M_0 t_i}, \\
\alpha_i &:= (g_{2M_0 t_i+1}, \ldots, g_{2M_0 t_i + M_0}), \\
\beta_i &:= (g_{2M_0 t_i + M_0 + 1}, \ldots, g_{2M_0 t_i + 2M_0}), \\
v_i &:= g_{2M_0 t_i + 2M_0 + 1} \cdots g_{2M_0 t'_i}, \\
v_i^{(core)} &:= g_{2M_0 t_i + 2M_0 + 1} \cdots g_{2M_0 t_i + 2M_0 + 2M_0\, K_{sleep}}, \\
v_i^{(tail)} &:= g_{2M_0 t_i + 2M_0 + 2M_0\, K_{sleep} + 1} \cdots g_{2M_0 t'_i}, \\
\gamma_i &:= (g_{2M_0 t'_i + 1}, \ldots, g_{2M_0 t'_i + M_0}), \\
\delta_i &:= (g_{2M_0 t'_i + M_0 + 1}, \ldots, g_{2M_0 t'_i + 2M_0})
\end{aligned}$$

for $i = 1, \ldots, \mathcal{N}(n')$; we also define $w_{\mathcal{N}(n')} = g_{2M_0[t'_{\mathcal{N}(n')}+1]+1} \cdots g_n$.

We first determine the values of ρ_j's. Observe that $\mathcal{N}(n')$ and $\{t_j, t'_j\}_j$ depend solely on $\{\rho_j\}_j$, and

$$\mathbb{P}\left(\mathcal{N}(n') \le (1 - 1/N_0)\frac{n}{2M_0\, K_{sleep}}\right) \le \frac{1}{K_{11}} e^{-K_{11} n'} \tag{7.3.4}$$

for some $K_{11} > 0$ due to our assumption $K_{sleep} \ge N_0(K_6+1)$ ([Gou21, Lemma 5.13]). We fix values of $\{\rho_j\}_j$ that makes Inequality 7.3.4 hold.

Meanwhile, Corollary 4.1.8 asserts that

$$\mathbb{P}(\# P_n \le (1 - 20/N_0)\mathcal{N}(n')) \le \frac{1}{K_{12}} e^{-K_{12} n'} \tag{7.3.5}$$

for some $K_{12} > 0$ that depends on N_0. Let $\mathcal{E}_n$ be an equivalence class made by the extended pivoting as in Lemma 4.2.2 that has pivotal times $\{i(1) < \ldots < i(m)\}$ where $m \ge (1 - 10/N_0)\mathcal{N}(n')$. We can then apply Lemma 7.3.1 once we determine the distribution of $d(o, v_i)$'s for $i = i(1), \ldots, i(m)$.

Note that regardless of the choice of $v_i^{(core)}$, Lemma 7.3.3 for ν asserts that $d(o, v_i o) \ge d(o, v_i^{(core)} o) - C$ outside a set of probability ϵ_2. This implies that $d(o, v_i o)$ dominates $B \cdot [d(o, v_i^{(core)} o) - C]$, where B is the Bernoulli distribution with $B = 1$ for probability $1 - \epsilon_2$ and $B = 0$ for probability ϵ_2, independent from $v_i^{(core)}$. If we denote this distribution by R, we have

$$\mathbb{E}\, R \ge (1 - \epsilon_2)\, \mathbb{E}_{\mu^{*2M_0\, K_{sleep}}}[d(o, go) - C] \ge (1 - \epsilon_2) 2N_0\, K_{sleep}\, \lambda' - C.$$

Moreover, Lemma 7.3.1 implies that

$$\mathbb{P}(d(o, \omega_n\, o) \le A | \mathcal{E}_n) \le \mathbb{P}(R_1 + \ldots + R_{(1-100/N_0)m} \le A) + e^{-n/K_{13}}$$

for any A some $K_{13} > 0$. Now the standard large deviation theory for the addition of real i.i.d. provides a constant $K_{14} > 0$ such that

$$\mathbb{P}\left(R_1 + \ldots + R_{(1-100/N_0)m} \le [(1 - 2\epsilon_2)2M_0\, K_{sleep}\, \lambda' - C]\left(1 - \frac{100}{N_0}\right) m \,\middle|\, \mathcal{E}_n\right) \le e^{-K_{14} m}. \tag{7.3.6}$$

Due to our choices that satisfy Equation 7.3.2 and 7.3.3, we also have

$$\begin{aligned}
(1 - 2\epsilon_2)2M_0\, K_{sleep}\, \lambda' - C &\ge (1 - 2\epsilon_2)2M_0\, K_{sleep}(K + \epsilon_1) - C \\
&\ge \frac{K}{1 - 200/N_0} \cdot 2M_0\, K_{sleep}\,.
\end{aligned} \tag{7.3.7}$$

We now combine Inequality 7.3.4, 7.3.5, 7.3.6 and 7.3.7 to deduce that

$$\mathbb{P}\left(d(o, \omega_n\, o) \le \frac{K}{1 - 200/N_0} 2M_0\, K_{sleep} \cdot \left(1 - \frac{100}{N_0}\right)\left(1 - \frac{20}{N_0}\right)\left(1 - \frac{1}{N_0}\right)\frac{n}{2M_0\, K_{sleep}}\right)$$

decays exponentially. This implies that $\mathbb{P}(d(o, \omega_n\, o) \le Kn)$ decays exponentially.

For the translation length and the BGIP of a random mapping class, we employ the estimation in Lemma 7.3.1. Namely, we have

$$\mathbb{P}(\omega_n \text{ does not have BGIP or } \tau(\omega_n) \le A|\mathcal{E}_n) \le \mathbb{P}(R_1 + \ldots + R_{(1-150/N_0)m} \le A) + e^{-n/K_{13}}$$

for some $K_{13} > 0$. Then by a similar reason as above, we deduce that

$$\mathbb{P}\left(\begin{array}{c}\omega_n \text{ does not have BGIP or} \\ \tau(\omega_n) \le \frac{K}{1-200/N_0} 2M_0\, K_{sleep} \cdot \left(1 - \frac{150}{N_0}\right)\left(1 - \frac{20}{N_0}\right)\left(1 - \frac{1}{N_0}\right)\frac{n}{2M_0\, K_{sleep}}\end{array}\right)$$

decays exponentially. □

7.4 Completion of Theorem C

We now explain the converse of CLT for translation length, hence finishing the proof of Theorem C.

Proposition 7.4.1. *Let μ be a non-elementary measure on* $\mathrm{Mod}(\Sigma)$ *with infinite second moment. Then for any sequence $(c_n)_n$ of real values, $\{\frac{1}{\sqrt{n}}[\tau(\omega_n) - c_n]\}_n$ does not converge in law.*

Proof. We continue from the proof of Proposition 7.4.1. Namely, we pinpoint subsets S_1, S_2 of S with cardinality $N_0/2$ such that $\mathbb{E}_{\mu'}[d(o, go)^2] = +\infty$, where μ' is the normalized restriction of μ on $A(S_1, S_2)$. Then we consider the decomposition

$$\mu^{(4M_0+1)} = \alpha(\mu_{S_1}^2 \times \mu' \times \mu_{S_2}^2) + (1-\alpha)\nu$$

for some $0 < \alpha < 1$ and ν and employ the first model described in Section 4.2; we have that

$$\mathbb{P}(\#\mathcal{P}_n(\omega) \le Kn) < Ke^{-n/K}$$

for some $K > 0$.

We now gather all sample paths with at least 2^{m+1} pivotal times till n, where $m = \lfloor \log_2 Kn \rfloor - 1$; this misses only a set of probability less than $K2^{-n/K}$. At the moment, we consider the usual pivoting (as in Section 4.1) at the first and the last 2^{m-2} pivotal times and the v-pivoting (as in Section 4.2) at the intermediate pivotal times to construct an equivalence class $\mathcal{E}$. On $\mathcal{E}$, we have $\omega \in S_k^{\dagger}$ holds for some $k \le 2^{m-1}$ with probability at least $1 - (8/N_0)^{2^{m-2}}$ by Lemma 7.2.2. We freeze such choices for the usual pivoting at the first and the last 2^{m-2} pivotal times, and freeze some more choices for the v-pivoting at some intermediate pivotal times, to leave the freedom of 2^m v-pivotal choices at the intermediate pivotal times $i(1) < \ldots < i(2^m)$. On the finer equivalence class $\mathcal{E}_1$ after this freezing, let us define x_i's as

$$x_{2^{m+1}k+2l-1} := \omega_{i(l)+2M_0}\, o, \;\; x_{2^{m+1}k+2l} := \omega_{i(l)+2M_0+M}\, o \quad (k \in \mathbb{Z}, l = 1, \ldots, 2^m).$$

Then as before, we have that $(x_i, x_k)_{x_j} \le E_0$ for all $i < j < k$ Here, note that $d(x_0, x_1) = d(x_{2^{m+1}}, x_{2^{m+1}+1}) = \ldots$ is constant over $\mathcal{E}_1$, since it only depends on the pivotal choices that we have already frozen. We have that

$$\tau(\omega_n) = \underbrace{\sum_{i=1}^{2^m} d(x_{2i-2}, x_{2i-1})}_{I_1} + \underbrace{\sum_{i=1}^{2^m} d(x_{2i-1}, x_{2i})}_{I_2} - 2\underbrace{\sum_{l=0}^{m}\sum_{k=1}^{2^{m-l}} (x_{2^l(2k-2)}, x_{2^l\cdot 2k})_{x_{2^l(2k-1)}}}_{I_3} + I_4,$$

where

$$I_4 := \lim_k \frac{1}{k} \sum_{l=1}^{k-1} (x_0, x_{(l+1)2^{m+1}})_{x_{l2^{m+1}}}$$

is bounded by E_0. Now the rest of the proof of Proposition 6.3.1 applies. $\square$

Chapter 8. Counting problem

The main purpose of this chapter is to prove the following:

Theorem 8.0.1 (Translation length grows linearly)**.** *For each $\lambda > 1$, there exists $\lambda_0 > 0$ such that the following holds. Let G be a finitely generated non-elementary subgroup of* $\mathrm{Mod}(\Sigma)$ *and $S' \subseteq G$ be a finite symmetric generating set. Then there exists a set $S'' \supseteq S'$ of G with $\#S'' \leq (1+\lambda)\#S' + \lambda_0$ such that*

$$\frac{\#\{g \in B_{S''}(n) : \tau_X(g) \leq Ln\}}{\#B_{S''}(n)} \leq Ke^{-n/K}$$

holds for some $L > K$.

Our strategy is to add Schottky isometries to S'. We encounter one technicality: the K_0-Schottky set S that we have in hand can never be symmetric, considering Lemma 3.2.10. Hence, in the following construction, we should allow choosing $\alpha_i, \beta_i, \gamma_i, \delta_i$ from $S \cup \check{S}$ for the pivotal time construction. Here, recall again that

$$\check{S} := \{s^{-1} : s \in S\} = \{(a_{M_0}^{-1}, \ldots, a_1^{-1}) : (a_1, \ldots, a_{M_0} \in S\}.$$

Lemma 8.0.2. *Let $s_i \in S$ and $\epsilon_i \in \{\pm 1\}$ for $i = 1, \ldots, k$. Suppose that there does not exist i such that $s_i = s_{i+1}$ and $\epsilon_i \epsilon_{i+1} = -1$. Then:*

1. *the sequence*

$$\left(\Gamma(s_1^{\epsilon_1}), \quad \Pi(s_1^{\epsilon_1})\Gamma(s_2^{\epsilon_2}), \quad \ldots, \quad \Pi(s_1^{\epsilon_1}) \cdots \Pi(s_{k-1}^{\epsilon_k})\Gamma(s_k^{\epsilon_k})\right)$$

   is D_0-aligned, and

2. *$\Pi(s_1^{\epsilon_1}) \cdots \Pi(s_k^{\epsilon_k})$ is not the identity element.*

Proof. This is a variant of Lemma 3.2.10. Recall again that

$$\mathrm{diam}\left(\pi_{\Gamma(s^\epsilon)}(\Pi(s^\epsilon)o) \cup o\right) = \mathrm{diam}\left(\Pi(s^\epsilon)o \cup o\right) \geq M_0/K_0 - K_0 > K_0$$

holds for each $s \in S$ and $\epsilon \in \{\pm 1\}$. This implies that

$$\mathrm{diam}\left(\pi_{\Gamma^n(s')}(\Pi(s^\epsilon)o) \cup o\right) \leq K \tag{8.0.1}$$

holds for all n if $s \neq s'$ (Property (2)), and for $n\epsilon \leq 0$ if $s = s'$ (Property (3)).

Now for each i, we have the following cases.

1. $s_i \neq s_{i+1}$: then we have

$$\mathrm{diam}\left(\pi_{\Gamma\left(s_{i+1}^{\epsilon_{i+1}}\right)}(\Pi(s_i^{-\epsilon_i})o) \cup o\right) \leq K_0, \quad \mathrm{diam}\left(\pi_{\Gamma(s_i^{\epsilon_i})}(\Pi(s_i^{\epsilon_i})o) \cup \Pi(s_i^{\epsilon_i})o\right) = 0.$$

   Here, the first inequality is Inequality 8.0.1 and the second inequality is immediate. Hence, $\left(\Gamma(s_i^{\epsilon_i}), \Pi(s_i^{\epsilon_i})\Gamma(s_{i+1}^{\epsilon_{i+1}})\right)$ is D_0-aligned by Lemma 3.1.2.

2. $s_i = s_{i+1}$: then $\epsilon_i = \epsilon_{i+1}$, and the above inequalities similarly hold.

This concludes the D_0-alignment. Now the nontriviality of $\Pi(s_1^{\epsilon_1}) \cdots \Pi(s_k^{\epsilon_k})$ follows from this D_0-alignment, namely,

$$d(o, \Pi(s_1^{\epsilon_1}) \cdots \Pi(s_k^{\epsilon_k})o) \geq \left[\sum_{i=1}^{k} d\left(o, \Pi(s_k^{\epsilon_k})o\right)\right] - 2(k-1)E_0 \geq E_0 k. \qquad \square$$

This leads to the following corollary:

Corollary 8.0.3. *S and $\check{S}$ are disjoint. Moreover, if we define*

$$T := \{(s_1, s_2, s_3, s_4) \in \left(s_i \in S \cup \check{S}\right)^4 : s_i \neq s_{i+1}^{-1} \text{ for } i = 1, 2, 3\} \tag{8.0.2}$$

and the map

$$\Phi : T \to G, \quad \Phi(s_1, s_2, s_3, s_4) := \Pi(s_1)\Pi(s_2)\Pi(s_3)\Pi(s_4), \tag{8.0.3}$$

then f is injective.

Proof of Theorem 8.0.1. Let us first observe the function

$$f(x) := \frac{1}{1+\sqrt{\lambda}} \left(\frac{\sqrt{\lambda}}{x}\right)^x \left(\frac{1}{1-x}\right)^{1-x}.$$

We have $\lim_{x\to 0+} f(x) = 1/(1+\sqrt{\lambda}) < 0.5$ so there exists $0 < \epsilon_1 < 1/3$ such that $f(\epsilon_1) \leq 1/2$. We then set

$$\lambda_0 = \left\lceil \left(24\sqrt{\lambda}\right)^4 \left(2^{20/\epsilon_1} + \left(\frac{1}{\sqrt{\lambda}-1}\right)^4\right)\right\rceil.$$

Our choice of λ_0 satisfies that:

$$1 - \frac{12}{\sqrt[4]{\lambda_0}} \geq 1 - \frac{1}{2\sqrt{\lambda}/(\sqrt{\lambda}-1)} = \frac{1+1/\sqrt{\lambda}}{2} \geq 1/\sqrt{\lambda}, \tag{8.0.4}$$

$$\lambda_0/\sqrt{\lambda} \geq 12 \cdot 2^{20/\epsilon_1}. \tag{8.0.5}$$

Given S', let S be the K_0-Schottky set with cardinality $\lfloor \frac{1}{2}\sqrt[4]{\lambda \# S' + \lambda_0}\rfloor$. We then define T and $\Phi : T \to G$ as in Equation 8.0.2 and 8.0.3. We then have

$$N_0 := \#\Phi(T) = \#T = \left(2\left\lfloor \frac{1}{2}\sqrt[4]{\lambda\#S' + \lambda_0}\right\rfloor\right)\left(2\left\lfloor \frac{1}{2}\sqrt[4]{\lambda\#S' + \lambda_0}\right\rfloor - 1\right)^3 \leq \lambda\#S' + \lambda_0$$

and

$$\begin{aligned} N_0 &\geq \left(\sqrt[4]{\lambda\#S' + \lambda_0} - 3\right)^4 \geq (\lambda\#S' + \lambda_0)\left(1 - \frac{12}{\sqrt[4]{\lambda\#S' + \lambda_0}}\right) \\ &\geq \sqrt{\lambda}\#S' + \lambda_0/\sqrt{\lambda} \geq \sqrt{\lambda}\#S' + 8 \cdot 2^{20/\epsilon_1}. \end{aligned}$$

Here, we used Inequality 8.0.4 and 8.0.5 at the second and the third inequalities, respectively.

We consider the simple random walk on $S' \cup \Phi(T)$. We have

$$\mu = \alpha\mu_{\Phi(T)} + (1-\alpha)\nu,$$

where $\mu_{\Phi(T)}$ is the uniform measure on $\Phi(T)$ and ν is the uniform measure on the remaining choices. Here, note that $\alpha \geq \sqrt{\lambda}/(1+\sqrt{\lambda})$. This decomposition enables us to employ the model in Section 4.3. Namely, we define Bernoulli RVs ρ_i with expectation α, η_i' with the law $\mu_{\Phi(T)}$ and ν_i with the law ν, all independent, and define $g_{i+1} = \eta_i'$ when $\rho_i = 1$ and $g_{i+1} = \nu_i$ otherwise. Then $(g_i)_{i=1}^{\infty}$ has the law μ^{∞}. We define $\mathcal{N}(k) := \sum_{i=0}^{k} \rho_i$ and $\vartheta(i) := \min\{j \geq 0 : \mathcal{N}(j) = i\}$ as before.

Let us now estimate the probability that $\mathcal{N}(n-1) \leq \epsilon_1 n$. Since $\mathcal{N}(n-1)$ is greater in distribution than the sum of n independent Bernoulli distribution with expectation $\sqrt{\lambda}/(1+\sqrt{\lambda})$, we have

$$\mathbb{P}\left(\mathcal{N}(n-1) \leq \epsilon_1 n\right) \leq \sum_{i=0}^{\epsilon_1 n} \binom{n}{i} \left(\frac{\sqrt{\lambda}}{1+\sqrt{\lambda}}\right)^i \left(\frac{1}{1+\sqrt{\lambda}}\right)^{n-i}.$$

Since $\epsilon_1/(1-\epsilon_1) \leq 1/\sqrt{\lambda}$, the term $a_i = \binom{n}{i}\left(\frac{\sqrt{\lambda}}{1+\sqrt{\lambda}}\right)^i \left(\frac{1}{1+\sqrt{\lambda}}\right)^{(1-\epsilon_1)n}$ is monotonically increasing for $i = 0, \ldots, \epsilon_1 n$. Hence, the probability is bounded by

$$\epsilon_1 n \cdot \binom{n}{\epsilon_1 n}\left(\frac{\sqrt{\lambda}}{1+\sqrt{\lambda}}\right)^{\epsilon_1 n}\left(\frac{1}{1+\sqrt{\lambda}}\right)^{(1-\epsilon_1)n}.$$

The growth rate of this term is $f(\epsilon_1)$, which is smaller than $1/2$. Hence, we can conclude that

$$\mathbb{P}\left(\mathcal{N}(n-1) \leq \epsilon_1 n\right) \leq \frac{C}{2^n}$$

for some $C > 0$.

Now given the choices of $\{\rho_i\}_i$ that gives $\mathcal{N}(n-1) \geq \epsilon_1 n$, we further fix the values of ν_i's. At the moment, we define $\eta_i := (\alpha_i, \beta_i, \gamma_i, \delta_i) \in T$ such that $\Phi(\alpha_i, \beta_i, \gamma_i, \delta_i) = \eta'_{\vartheta(i)+1}$; note that the correspondence $\eta_i \leftrightarrow \eta'_{\vartheta(i)+1}$ is one-to-one. Then η_i's are chosen with the uniform measure $T^{\mathcal{N}(n-1)}$. Following the convention in Section 4.3, we define

$$w_i := g_{\vartheta(i-1)+2} \cdots g_{\vartheta(i)}.$$

We also set $w^{(n)} := g_{\vartheta(\mathcal{N}(n-1))+2} \cdots g_n$ and $a_i = \Pi(\alpha_i), \ldots, d_i = \Pi(\delta_i)$. Then we have

$$\begin{aligned}
\omega_n &= w_0 \nu_{\vartheta(1)} w_2 \cdots \nu_{\vartheta(\mathcal{N}(n-1))} w^{(n)} \\
&= w_0 a_1 b_1 c_1 d_1 w_1 \cdots a_{\mathcal{N}(n-1)} b_{\mathcal{N}(n-1)} c_{\mathcal{N}(n-1)} d_{\mathcal{N}(n-1)} w^{(n)}.
\end{aligned}$$

In this setting, we define the set of pivotal times as in Section 4.1 (with $v_i = id$ identically). Since $(\alpha_i, \beta_i, \gamma_i, \delta_i)_{i=1}^{\mathcal{N}(n-1)}$ is chosen with the uniform measure on $T^{\mathcal{N}(n-1)}$, not $S^{\mathcal{N}(n-1)}$, the following modifications are needed.

- For choices $(\alpha_i, \beta_i, \gamma_i, \delta_i) \in T$, Observation 4.1.2 still holds thanks to Lemma 8.0.2.
- In Lemma 4.1.4, we first pick $\delta_i \in S \cup \check{S}$, and then $\gamma_i \in (S \cup \check{S}) \setminus \{\delta_i\}^{-1}$, and then $\beta_i \in (S \cup \check{S}) \setminus \{\gamma_i^{-1}\}$, and then $\alpha_i \in (S \cup \check{S}) \setminus \{\beta_i^{-1}\}$. First, there exists at most 1 candidate for δ_k that violates Condition 4.1.3; this rules out at most $(N_0-1)^3$ choices in T. Picking δ_k that satisfies Condition 4.1.3, Condition 4.1.2 and 4.1.4 are automatically guaranteed for any valid γ_k and β_k due to the definition of T and Lemma 8.0.2. Finally, there exists at most 1 candidate for α_k that violates Condition 4.1.5. This rules out at most $N_0(N_0-1)^2$ choices in T. Overall, we have
$$\mathbb{P}\Big(\#P_k(s, \alpha_k, \beta_k, \gamma_k, \delta_k) = \#P_{k-1}(s) + 1\Big) \geq 1 - \frac{(2N_0-1)(N_0-1)^2}{N_0(N_0-1)^3} \geq 1 - \frac{2}{N_0-1}.$$
- Let us investigate the proof of Lemma 4.1.6. We first have
$$\mathbb{P}(\mathcal{A}|T) \geq 1 - \frac{2}{N_0-1}.$$
Next, in the case of $j=1$ we similarly set $l < m$ as the last 2 elements of $P_{k-1}(s)$. Fixing $(\alpha_k, \beta_k, \gamma_k, \delta_k) \in T$ and $\tilde{s} \in \mathcal{E}_{k-1}(s)$, we define $\tilde{A} = \tilde{A}(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) \in \tilde{S}_m(s)$ as in the proof of Lemma 4.1.6. In other words, for $(\tilde{\alpha}_m, \bar{\beta}_m, \tilde{\gamma}_m) \in \tilde{A}$, $\bar{\beta}_m$ is now subject to Condition 4.1.7 in addition to the standing condition that $\bar{\beta}_m \neq \tilde{\alpha}_m^{-1}, \tilde{\gamma}_m^{-1}$. Since the additional Condition 4.1.7 rules out at most 1 choice, we have the conditional expectation
$$\frac{\#[E(\tilde{s}, \tilde{S}_m) \setminus E(\tilde{s}, \tilde{A})]}{\#E(\tilde{s}, \tilde{S}_m)} \leq \frac{1}{N_0-2} \geq \frac{2}{N_0-1}.$$

This leads to the estimation

$$\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - 1 \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s), (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right)$$
$$\leq \frac{2}{N_0 - 1} \cdot \frac{2}{N_0 - 1}.$$

By similar induction steps, we get

$$\mathbb{P}\left(\#P_k(\tilde{s}, \alpha_k, \beta_k, \gamma_k, \delta_k) < \#P_{k-1}(s) - j \,\middle|\, \tilde{s} \in \mathcal{E}_{k-1}(s),\, (\alpha_k, \beta_k, \gamma_k, \delta_k) \in S^4\right) \leq \left(\frac{2}{N_0 - 1}\right)^{j+1}$$

- For the pivoting for translation length, let us compare the proportion of $S_k^\dagger$ in $\tilde{S}^*_{i(k)} \times \tilde{S}^*_{i(M-k+1)}$ for an equivalence class $\mathcal{E}$ with M pivotal times. Fixing valid choices for $\beta_{i(k)}, \gamma_{i(k)}, \alpha_{i(M-k+1)}, \gamma_{i(M-k+1)}$, we now have three constraints for $\alpha_{i(k)}$: $\alpha_{i(k)} \neq \beta_{i(k)}^{-1}$, Condition 4.1.5 and Condition 7.2.1. In other words, among at least $N_0 - 2$ choices of $\alpha_{i(k)}$ that makes $(\alpha_{i(k)}, \beta_{i(k)}, \gamma_{i(k)}) \in \tilde{S}^*_{i(k)}$, all choices but at most one satisfy Condition 7.2.1. Fixing such $\alpha_{i(k)}$, we obtain a similar estimate for $\beta_{i(M-k+1)}$ and we conclude

$$\mathbb{P}\left(\alpha_{i(k)} \in S_k^\dagger(s), \beta_{i(M-k+1)} \in S^\dagger_{M-k+1}(s) \text{ for some } k \leq m \,\middle|\, \mathcal{E}\right) \geq 1 - \left(\frac{2}{N_0 - 2}\right)^m.$$

Having these modifications, we now estimate

$$\mathbb{P}\left(\#P_n(\omega) \geq \epsilon_1 n/2 \,\middle|\, \mathcal{N}(\omega_n) \geq \epsilon_1 n\right).$$

If $\mathcal{N}(\omega_n) = N$, then $\#P_n(\omega)$ is greater in distribution than the sum of N i.i.d. X_i with the distribution

$$\mathbb{P}(X_i = j) = \begin{cases} 1 - \frac{2}{N_0 - 1} & \text{if } j = 1, \\ \left(1 - \frac{2}{N_0-1}\right)\left(\frac{2}{N_0-1}\right)^{-j} & \text{if } j < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{8.0.6}$$

Note that

$$\mathbb{E}\left[\sqrt{\frac{2}{N_0 - 1}}^{X_i}\right] = \left(1 - \frac{2}{N_0 - 1}\right)\left[\sqrt{\frac{2}{N_0 - 1}} + \sum_{i=1}^{\infty} \sqrt{\frac{2}{N_0 - 1}}^{i}\right] \leq 2.1\sqrt{\frac{2}{N_0 - 1}}.$$

We then calculate:

$$\mathbb{P}\left(\sum_{i=1}^N X_i < \epsilon_1 n/2\right) \cdot \sqrt{\frac{2}{N_0 - 1}}^{\epsilon_1 n/2} \leq \mathbb{E}\left[\sqrt{\frac{2}{N_0 - 1}}^{\sum_{i=1}^N X_i}\right] = \prod_{i=1}^N \mathbb{E}\left[\sqrt{\frac{2}{N_0 - 1}}^{X_i}\right] \leq 2.1^{\epsilon_1 n} \cdot \sqrt{\frac{2}{N_0 - 1}}^{\epsilon_1 n}.$$

This implies that

$$\mathbb{P}\left(\sum_{i=1}^N X_i < \epsilon_1 n/2\right) \leq 2.1^{\epsilon_1 n} \cdot \left(\frac{2}{N_0 - 1}\right)^{\epsilon_1 n/4} \leq \left(\frac{2 \cdot 20}{N_0}\right)^{\epsilon_1 n/4} \leq \frac{1}{2^n}.$$

At the final stage we used $N_0 \geq 40 \cdot 2^{5/\epsilon_1}$.

Now, for an equivalence class $\mathcal{E}_n$ with $P_n(\mathcal{E}_n) \geq \epsilon_1 n/2$, we know that ω is BGIP with $\tau(\omega) \geq \epsilon_1 n/10$ except probability

$$\left(\frac{2}{N_0 - 2}\right)^{\epsilon_1 n/5} \leq \frac{1}{2^n}.$$

In summary, $\mathbb{P}(\omega_n$ is not BGIP or $\tau(\omega_n) \geq \epsilon_1 n/10) \leq 1 - (1/2)^n$; the number of sample paths corresponding to this event is at most $((\#S' + N_0)/2)^n$.

Meanwhile, the ball $B_n(e)$ contains all

$$\{\Pi(s_1)\cdots\Pi(s_{4n}) : s_i \in S_0, s_i \neq s_{i+1}^{-1}\}.$$

Their number is at least

$$(\sqrt[4]{\lambda\#S' + \lambda_0} - 3)^{4n} \geq \left((\lambda\#S' + \lambda_0)\left(1 - \frac{12}{\sqrt[4]{\lambda_0}}\right)\right)^n \geq \left((\lambda\#S' + \lambda_0)\left(\frac{1 + 1/\sqrt{\lambda}}{2}\right)\right)^n.$$

Since

$$\#S' + N_0 \leq (1+\lambda)\#S' + \lambda_0 \leq (\sqrt{\lambda} + \lambda)\#S' + \lambda_0(1 + 1/\sqrt{\lambda}),$$

we conclude that the growth rate of $\#B_n(e)$ is strictly greater than the growth rate of elements w in $\#B_n(e)$ such that w is not BGIP or $\tau(w) \geq \epsilon_1 n/10$. $\square$

Chapter 9. Spaces with contracting isometries

We discuss spaces other than Teichmüller space that possess BGIP isometries. At least two important spaces arise in geometric group theory and geometric topology: CAT(0) spaces and Outer space.

9.1 CAT(0) spaces

CAT(0) spaces are geodesic spaces where geodesic triangles are not fatter than Euclidean triangles with the same side lengths. A typical example is a complete Riemannian manifold with non-positive sectional curvature. In particular, Euclidean spaces and n-dimensional hyperbolic spaces are CAT(0) spaces, and products of CAT(0) spaces (with l^2 metric) are also CAT(0).

Definition 9.1.1 (Isometries of CAT(0) spaces)**.** *Let X be a CAT(0) space. An isometry g of X is said to be* semisimple *if $x \mapsto d(x, gx)$ attains its minimum in X. If the minimum is 0, then g is said to be* elliptic*. Otherwise, g is said to be* axial*. g is said to be* rank-1 *if it is axial and its axis is strongly contracting.*

In view of Lemma 2.2.6, rank-1 isometries have BGIP. In the setting of proper CAT(0) spaces, the following propositions guarantee the existence of non-elementary subgroups of $\mathrm{Isom}(X)$.

Proposition 9.1.2 ([BF09, Theorem 5.4])**.** *Let X be a proper CAT(0) space. Then an axial isometry g is rank-1 if and only if its axis does not bound a flat half-plane.*

Proposition 9.1.3 ([BF09, Theorem 6.5])**.** *Let X be a proper CAT(0) space. Suppose that the action of $\Gamma \le \mathrm{Isom}(X)$ satisfies WPD (see [BF09, Definition 6.4]). Then Γ is non-elementary.*

Proposition 9.1.4 ([Ham09, Corollary 5.4])**.** *Let X be a proper CAT(0) space that admits a rank-1 isometry. Suppose that the limit set of $\mathrm{Isom}(X)$ on the visual boundary has at least 3 points and $\mathrm{Isom}(X)$ does not globally fix a point in ∂X. Then $\mathrm{Isom}(X)$ is non-elementary.*

Proposition 9.1.5 ([CF10, Proposition 3.4])**.** *Let X be a proper CAT(0) space that admits a rank-1 isometry. Suppose that $\mathrm{Isom}(X)$ does not globally fix a point in ∂X nor stabilize a geodesic line. Then $\mathrm{Isom}(X)$ is non-elementary.*

As mentioned in Section 2.3, the Weil-Petersson metric is an (incomplete) CAT(0) metric on Teichmüller space.

9.2 CAT(0) cube complices

It is expected that many irreducible CAT(0) spaces (i.e., those that are not products of two spaces) contain a rank-1 isometry. In particular, Ballmann and Buyalo conjectured in [BB08] the following: if X is an irreducible, locally compact, complete CAT(0) space and Γ is an infinite discrete group acting properly and cocompactly on X, then either X is a higher-rank symmetric space, is a Euclidean building of higher dimension, or has a rank-1 isometry. Although this conjecture is not settled in full generality, it has been established for the following class of spaces called CAT(0) cube complices by Caprace and Sageev.

Proposition 9.2.1 ([CS11, cf. Theorem A]). *Let X be an irreducible, finite-dimensional CAT(0) cube complex and $\Gamma \leq \mathrm{Aut}(X)$ be a group that does not globally fix a point nor stabilize a 1-dimensional flat in $X \cup \partial_\infty X$. Then X contains a convex Γ-invariant subcomplex Y on which the action of Γ is non-elementary.*

This follows by combining Theorem A, Theorem E, Lemma 6.2, Lemma 7.1 and the Double Skewering Lemma in [CS11].

We now give a remark on two different metrics on CAT(0) cube complices. Traditionally, CAT(0) cube complices come equipped with either the metric induced by gluing Euclidean cubes or the metric induced by gluing cubes with l^1-metrics. The first metric is referred to as the l^2-metric or the CAT(0) metric, and the second metric is referred to as the l^1-metric or the combinatorial metric. In finite-dimensional CAT(0) cube complices, these two metrics are quasi-isometric. The previous theorem was with respect to the l^2-metric, but one can also discuss the same result with respect to the l^1-metric. More precisely, the proof of Proposition 9.2.1 guarantees the existence of the following objects:

- half-spaces $\mathfrak{h}_1'$, $\mathfrak{h}_1''$, $\mathfrak{h}_2'$, $\mathfrak{h}_2''$ such that $\mathfrak{h}_1'$, $\mathfrak{h}_1'^*$, $\mathfrak{h}_2'$, $\mathfrak{h}_2'^*$ are mutually strongly separated;
- automorphisms g_1 that sends $\mathfrak{h}_1'$ to $g_1\mathfrak{h}_1' \supseteq \mathfrak{h}_1''$ and g_2 that sends $\mathfrak{h}_2'$ to $g_2\mathfrak{h}_2' \supseteq \mathfrak{h}_2''$.

Then by a classic ping-pong lemma, one can construct translates of the ones associated with $\mathfrak{h}_1'$, $\mathfrak{h}_2'$ in between $g_1^{\pm n}o$ and $g_2^{\pm m}o$, whose number increases as $n, m \to +\infty$. Moreover, using Lemma 3.5 of [CFI16] instead of Lemma 6.1 of [CS11] implies that g_1, g_2 are strongly contracting. Hence we deduce that g_1, g_2 are independent strongly contracting isometries with respect to the l^1-metric also.

We also note the characterization of contracting isometries (with respect to the l^1-metric) of (not necessarily finite-dimensional) CAT(0) cube complices by Genevois [Gen20]. In the same paper, Genevois also detects contracting isometries of a locally finite CAT(0) cube complex from the structure of the so-called combinatorial boundary.

An important family of examples comes from right-angled Artin groups (RAAGs). Recall that a RAAG Γ is associated with a simply connected CAT(0) cube complex $\tilde{X}_\Gamma$; Γ acts properly and cocompactly on $\tilde{X}_\Gamma$, and the resulting quotient is called the *Salvetti complex* X_Γ of Γ. It is proved in [BC12] that if Γ is not a direct product, then the universal cover $\tilde{X}_\Gamma$ of the Salvetti complex admits a rank-1 isometry. This is proved by finding $g \in \Gamma$ that has infinite join length, or equivalently, infinite separation length. This implies that a suitable power of g serves as a double skewer; given the previous discussion, g is strongly contracting with respect to the l^1-metric also. Moreover, since the action of Γ on X_Γ is properly discontinuous, one can employ Proposition 9.1.3 and conclude that the action of Γ on X_Γ is non-elementary with respect to both l^2-metric and l^1-metric.

9.3 Outer space and the Lipschitz metric

In this subsection, we gather facts regarding the outer automorphism group and Outer space. For detailed definitions and theories, see the general exposition of Vogtmann [Vog15] or individual papers, e.g. [BH92], [FM11], [FM12], [AKB12], [AK11], [DT18] and [KMPT22].

Let X be the Culler-Vogtmann Outer space CV_N of rank $N \geq 3$, which is the space of unit-volume marked metric graphs with fundamental group F_N. In other words, a point $p \in CV_N$ corresponds to the homotopic class of a homotopy equivalence $h : R_N \to \Gamma$, where R_N is a fixed rose with N petals and Γ is a unit-volume metric graph. The corresponding space without the volume normalization is called the unprojectivized Outer space cv_N, and there is a projectivization from cv_N to CV_N by dilation.

Outer space comes equipped with a canonical metric, the Lipschitz distance, which is defined as follows: for two markings $h_1 : R_N \to \Gamma_1$ and $h_2 : R_N \to \Gamma_2$, the distance from Γ_1 to Γ_2 is defined by

$$d_{CV}(\Gamma_1, \Gamma_2) := \inf\{\log \mathrm{Lip}(f) : f \sim f_2 \circ f_1^{-1}\},$$

where $\mathrm{Lip}(f)$ is the (maximal) Lipschitz constant of f. *We now make a convention that differs from the traditional one.* Namely, the outer automorphism group $\mathrm{Out}(F_N)$ of rank N acts on CV_N by changing the basis of the marking *with the inverses*: given $\phi \in \mathrm{Out}(F_N)$ and $h : R_N \to \Gamma$ representing a point of CV_N, ϕ moves h to $h \circ \phi^{-1} : F_N \overset{\phi^{-1}}{\to} F_N \overset{h}{\to} \Gamma$. This is a left action by isometries. We denote action by $X \ni h \mapsto \phi \cdot h \in X$.

It is known that the Lipschitz distance is asymmetric [FM11] and not uniquely geodesic. However, distances among ϵ-thick points (i.e., those with systole at least ϵ) have the coarse symmetry: there exists a constant $C = C(\epsilon) < +\infty$ such that for any ϵ-thick points x and y, one has $d(x, y) \le Cd(y, x)$ [AKB12]. In particular, distances among the translates of the reference point o by $\mathrm{Out}(F_N)$ satisfy the coarse symmetry.

Just as Teichmüller space $\mathcal{T}(\Sigma)$ is accompanied by the curve complex $\mathcal{C}(\Sigma)$ and the coarse projection $\pi^{\mathcal{C}} : \mathcal{T}(\Sigma) \to \mathcal{C}(\Sigma)$, CV_N is accompanied by the complex of free factors $\mathcal{FF}_N$ and the coarse projection $\pi^{\mathcal{FF}} : CV_N \to \mathcal{FF}_N$. This projection is coarsely $\mathrm{Out}(F_N)$-equivariant and coarsely Lipschitz. Moreover, geodesics in CV_N projects to K-unparametrized bi-quasigeodesics for some uniform $K > 0$ [BF14, Proposition 9.2].

Outer space also accommodates lots of BGIP isometries. We say that an outer automorphism $\phi \in \mathrm{Out}(F_N)$ is *reducible* if there exists a free product decomposition $F_N = C_1 * \cdots * C_k * C_{k+1}$, with $k \ge 1$ and $C_i \ne \{e\}$, such that ϕ permutes the conjugacy classes of $C_1, \ldots, C_k$. If not, we say that ϕ is *irreducible*. We also say that ϕ is *fully irreducible (or iwip)* if no power of ϕ is reducible, or equivalently, no power of ϕ preserves the conjugacy class of any proper free factor of F_N. We also say that ϕ is *atoroidal (or hyperbolic)* if no power of ϕ fixes any nontrivial conjugacy class in F_N. When ϕ is fully irreducible, it is non-atoroidal if and only if it is *geometric*, i.e., induced by a pseudo-Anosov $\varphi : \Sigma \to \Sigma$ on a compact surface Σ with one boundary component, via identification of F_N with $\pi(\Sigma)$. Bestvina and Feighn proved in [BF14] that $\phi \in \mathrm{Out}(F_N)$ is fully irreducible if and only if it acts on $\mathcal{FF}_N$ loxodromically.

We say that a subgroup $G \le Out(F_N)$ is *non-elementary* if it acts on $\mathcal{FF}_N$ in a non-elementary way, or equivalently, contains two fully irreducibles with mutually distinct attracting/repelling trees. It is known that if $G \le Out(F_N)$ does not fix any finite subset of $\mathcal{FF}_N \cup \partial\mathcal{FF}_N$, or equivalently, if it is not virtually cyclic nor virtually fixes the conjugacy class of a proper free factor of F_N, then G is non-elementary [Hor16]. Since $\pi^{\mathcal{FF}}$ is coarsely Lipschitz, the independence of two fully irreducibles in $\mathcal{FF}_N$ is lifted to the independence in CV_N.

We refer the readers to [BH92], [AK10], [BF14] and [AKKP19] for the precise definition of a train-track representative $f : \Gamma \to \Gamma$ of an outer automorphism ϕ. Roughly speaking, a train-track representative of ϕ is a self-map $f : \Gamma \to \Gamma$ in the free homotopy class of ϕ on a simplicial graph Γ that sends vertices to vertices, restricts to immersion on each edge of Γ and sends edges to immersed segments after iterations. It is due to Bestvina and Handel [BH92] that every irreducible outer automorphism admits a train-track representative, although it may not be unique.

Given such a structure, one can endow Γ with a metric such that f stretches each edge of Γ by the same constant $\lambda > 1$, which is called the *expansion factor* of f. This expansion factor is uniquely determined by the choice of ϕ and does not depend on the choice of f. Moreover, in view of Skora's interpretation of Stallings fold decompositions, one obtains a continuous path on cv_N from Γ to $\Gamma \circ \phi$

by folding a single illegal turn at each time (cf. [AKKP19]). This descends to a geodesic segment of length $\log \lambda$ (after a reparametrization) and the concatenation of its translates by powers of ϕ becomes a bi-infinite, ϕ-periodic geodesic. We call this a *(optimal) folding axis* of ϕ. Algom-Kfir observed the following:

Theorem 9.3.1 ([AK11]). *Folding axes of fully irreducible outer automorphisms are strongly contracting.*

Unfortunately, we need BGIP instead of the strongly contracting property in our setting, and the author does not know a way to promote the latter to the former. Meanwhile, I. Kapovich, Maher, Pfaff and Taylor observed the following version of BGIP in Outer space. This requires the notion of greedy folding paths, whose accurate definition can be found in [FM11], [BF14] and [DH18]. In short, a greedy folding path $\gamma : I \to cv_N$ is obtained by folding every illegal turn at each time with speed 1, where the illegal turn structures at different forward times are identical and define a well-defined illegal turn structure. This also descends to a geodesic on CV_N, and we have the following theorem:

Theorem 9.3.2 ([KMPT22, Theorem 7.8]). *Let $\phi \in \mathrm{Out}(F_N)$ be a fully irreducible outer automorphism. Suppose that γ is a bi-infinite, ϕ-periodic greedy folding path. Then there exists $C > 0$ such that the following holds.*

Let $x, y \in X$ be points such that $d^{sym}(\pi_\gamma(x), \pi_\gamma(y)) \geq C$, and satisfy $d^{sym}(\pi_\gamma(x)) = \gamma(t_1)$, $d^{sym}(\pi_\gamma(y)) = \gamma(t_2)$ for some $t_1 < t_2$. Then any geodesic $[x, y]$ between them contains a subsegment $[z_1, z_2]$ such that

$$d^{sym}(z_1, \pi_\gamma(x)) < C, \quad d^{sym}(z_2, \pi_\gamma(y)) < C.$$

This uni-directional version of BGIP is designed for outer automorphisms that have an invariant greedy folding line. It seems not shown that all fully irreducibles have such a line. (The author thanks Sam Taylor for pointing this out.) Nonetheless, by adapting Dowdall-Taylor's idea and Kapovich-Maher-Pfaff-Taylor's proof of Theorem 9.3.2, we can obtain the following result. This proof was kindly informed by Sam Taylor.

Proposition 9.3.3. *Let $\varphi \in \mathrm{Out}(F_N)$ be a fully irreducible outer automorphism. Then the orbit $\{\varphi^i o\}_{i \in \mathbb{Z}}$ of o by φ is a BGIP axis.*

Proof. Before we begin, we recall the following facts regarding a geodesic δ-hyperbolic space Y.

1. (Morse property) A K-quasigeodesic and a geodesic with the same endpoints are within Hausdorff distance $K_2 = K_2(K, \delta)$.

2. The closest point projections onto a K-quasigeodesic and a geodesic on Y with the same endpoints are within distance $K_3 = K_3(K, \delta)$.

3. If the projections of $x, y \in Y$ to K-quasigeodesic γ contain $\gamma(s)$ and $\gamma(t)$, respectively, and $d(\gamma(s), \gamma(t)) > K_4 = K_4(K, \delta)$, then $[x, y]$ and $[x, \pi_\gamma(x)] \cup \gamma|_{[s,t]} \cup [\pi_\gamma(y), y]$ are within Hausdoff distance K_4.

4. If K-quasigeodesics γ, γ' are within Hausdorff distant K and the distance between starting points is at most K, then γ' crosses γ up to a constant $K_5 = K_5(K, \delta)$, i.e., γ and $\gamma' \circ \rho$ K_5-fellow travel for some orientation-matching reparametrization ρ.

Let T^+, T^- be the attracting and repelling trees of φ, respectively. There exist optimal greedy folding lines $\gamma^\pm : \mathbb{R} \to CV_N$ such that

$$\lim_{t \to +\infty} \gamma^\pm(t) = T^\pm, \quad \lim_{t \to -\infty} \gamma^\pm(t) = T^\mp \tag{9.3.1}$$

([BR15], Lemma 6.7 and Lemma 7.3). Since $\{\varphi_i o\}_i$ is a quasigeodesic whose endpoints agree with γ^+, Theorem 4.1 of [DT18] asserts that $d_H(\{\varphi^i o\}_i, \gamma^+) < K_1$ and $\pi^{\mathcal{FF}}(\gamma^+)$ is a K_1-quasigeodesic for some K_1. Similarly, by comparing $\{\varphi_{-i} o\}_i$ and γ^-, we deduce that $d_H(\{\varphi^i o\}_i, \gamma^-) < K_1$ and $\pi^{\mathcal{FF}}(\gamma^-)$ is a K_1-quasigeodesic. Also, $\gamma^\pm$ are uniformly thick.

Let us now take $x_i^+ \in \pi_{\gamma^+}(\varphi^i o)$ and $x_i^- \in \pi_{\gamma^-}(\varphi^i o)$ for each i. We recall the following result of Dahmani and Horbez ([DH18, Proposition 5.17, Corollary 5.22]; see also Section 7 of [KMPT22]): there exist $B, D > 0$ such that $\gamma^\pm$ are (B, D)-contracting at $x_i^\pm$'s (with a suitable crossing constant κ). In other words, a geodesic η on CV_N projects to a path that κ-crosses up a large enough subsegment of $\pi^{\mathcal{FF}}\gamma^\pm$ that begins from $\pi^{\mathcal{FF}}(x_i^\pm)$, then η has a point p whose distance to $\gamma^\pm$ is bounded by D. Since $\gamma^\pm$ are thick, the distance from $\gamma^\pm$ to such point p is also controlled and η intersects a neighborhood of $\gamma^\pm$ in such a case.

We now observe that $\pi^{\mathcal{FF}}\pi_{\gamma^+}, \pi^{\mathcal{FF}}\pi_{\gamma^-}$ and $\pi_{\pi^{\mathcal{FF}}(\{\varphi^i o\}_i)} \circ \pi$ are coarsely equivalent. First, Lemma 4.11 of [DT18] asserts that $\pi_{\gamma^\pm}$ and $Pr_{\gamma^\pm}$ are equivalent, where Pr stands for the Bestvina-Feighn left projection. Then Lemma 4.2 of the same paper asserts that $\pi^{\mathcal{FF}} Pr_{\gamma^\pm}$ and $\pi_{\pi^{\mathcal{FF}}(\gamma^\pm)} \circ \pi$ are equivalent. These are then equivalent to $\pi_{\pi^{\mathcal{FF}}(\{\varphi^i o\}_i)} \circ \pi$, since $\pi^{\mathcal{FF}}(\gamma^\pm)$ and $\pi^{\mathcal{FF}}(\{\varphi^i o\}_i)$ are close to each other and $\pi^{\mathcal{FF}}(\{\varphi^i o\}_i)$, a quasi-geodesic on the Gromov hyperbolic space $\mathcal{FF}$, is strongly contracting.

We now lift these projections: we claim that π_{γ^+}, π_{γ^-} and $\pi_{\{\varphi^i o\}_i}$ are equivalent. First, suppose that $\pi_{\gamma^+}(x)$ and $\pi_{\gamma^-}(x)$ are far from each other for some $x \in X$. Since γ^+, γ^-, $\{\varphi^i o\}_i$ are close to each other, we may take $\varphi^i o$ and $\varphi^j o$ near $\pi_{\gamma^+}(x)$ and $\pi_{\gamma^-}(x)$, respectively, and conclude that $|i - j|$ is large. This implies that $\pi^{\mathcal{FF}}(\varphi^i o)$ and $\pi^{\mathcal{FF}}(\varphi^j o)$ are also far from each other (since φ is loxodromic on CV_N), and consequently $\pi^{\mathcal{FF}}(\pi_{\gamma^+}(x))$, $\pi^{\mathcal{FF}}(\pi_{\gamma^-}(x))$ are far from each other. $(*)$ Since we have proved that $\pi^{\mathcal{FF}}\pi_{\gamma^+}$ and $\pi^{\mathcal{FF}}\pi_{\gamma^-}$ are equivalent, this cannot happen. Hence, π_{γ^+} and π_{γ^-} are equivalent.

Now suppose that $\pi_{\{\varphi^i o\}_i}(x)$ and $\pi_{\gamma^\pm}(x)$ are far from each other for some $x \in X$. We take $\varphi^j o \in \pi_{\{\varphi^i o\}_i}(x)$ and $\varphi^{j'} o$ near $\pi_{\gamma^\pm}(x)$ and conclude that $|j' - j|$ is large. If j is much larger than j', then $\pi^{\mathcal{FF}}([x, \varphi^j o])$ is a quasigeodesic whose endpoints project onto $\pi^{\mathcal{FF}}(\{\varphi^i o\}_i)$ near $\pi^{\mathcal{FF}}\varphi^{j'} o$ and $\pi^{\mathcal{FF}}\varphi^j o$, respectively. Since $j' - j$ is large enough, this quasigeodesic crosses up long enough subsegments of $\pi^{\mathcal{FF}}(\{\varphi^i o\}_i)$ and $\pi^{\mathcal{FF}}(\gamma^+)$ that begin at $\pi^{\mathcal{FF}}(\varphi^{j'} o)$ and $\pi^{\mathcal{FF}}(x_{j'})$, respectively. Using the (B, D)-contraction at $x_{j'}^+$ of γ^+, we conclude that $[x, \varphi^j o]$ contains a point p nearby $x_{j'}^+$, which makes $d(x, \varphi^{j'} o)$ shorter than $d(x, \varphi^j o)$ and leads to a contradiction. Similar contradiction occurs due to the contracting property of γ^- at x_i^-'s when j' is much larger than j. Hence, $\pi_{\{\varphi^i o\}_i}(x)$ and $\pi_{\gamma^\pm}(x)$ are equivalent.

Now if a geodesic η on CV_N has a large projection on $\{\varphi^i o\}_i$, then it also has large projections on $\gamma^\pm$. This also forces large $\pi^{\mathcal{FF}}(\pi_{\gamma^\pm}(\eta))$, due to the argument as in $(*)$. When $\pi^{\mathcal{FF}}(\pi_{\gamma^\pm}(\eta))$ progresses in the forward direction with respect to $\{\varphi_i o\}_i$, then we employ the contracting property of γ^+ to conclude. If it progresses in the backward direction, then we employ the contracting property of γ^- to conclude. □

9.4 BGIP axes in asymmetric metric spaces

In this section, we prove Lemma 2.2.4, 2.2.5, 2.2.6, 2.2.7 and 2.2.8 for asymmetric metrics. Let us first fix the convention of asymmetric metrics.

Definition 9.4.1 (Metric space). *An* (asymmetric) metric space (X, d) *is a set* X *equipped with a function* $d : X \times X \to \mathbb{R}_{\geq 0}$ *that satisfies the following:*

- *for any* $x, y \in X$, $d(x, y) = 0$ *if and only if* $x = y$;
- *(triangle inequality) for any* $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$;

- *(local symmetry) for each $x \in X$, there exist $\epsilon, K > 0$ such that $d(y,z) \leq Kd(z,y)$ holds for $y, z \in \{a \in X : \min(d(x,a), d(a,x)) < \epsilon\}$.*

In this situation, we say that d is a metric *on X. d is said to be* symmetric *if $d(x,y) = d(y,x)$ holds for all $x, y \in X$. We define a symmetric metric called the* symmetrization *of d by*

$$d^{sym}(x,y) := d(x,y) + d(y,x).$$

We endow (X,d) with the topology induced by d^{sym}.

From now on, we fix a geodesic space X endowed with a possibly asymmetric metric. We begin with the following technical lemma.

Lemma 9.4.2. *Let γ be a K-bi-quasigeodesic such that $\pi_\gamma(y) \neq \emptyset$ for any $y \in X$. Let also $x \in \overline{\mathcal{N}_K(\gamma)}$. Then $d(x,p) \leq K$ and $d(p,x) \leq 3K^3 + 2K$ hold for any $p \in \pi_\gamma(x)$.*

Proof. Let us take $\epsilon > 0$ and $y \in \mathcal{N}_\epsilon(x) \cap \mathcal{N}_K(\gamma)$. For $p \in \pi_\gamma(x)$ and $q \in \gamma$ such that $d^{sym}(q,y) \leq K$, we observe

$$\begin{aligned}
d(x,p) &\leq d(x,q) \leq d(x,y) + d(y,q) \leq \epsilon + K, \\
d(p,x) &\leq d(p,q) + d(q,y) + d(y,x) \\
&\leq [K^2 d(q,p) + K^3 + K] + K + \epsilon \\
&\leq K^2[d(q,y) + d(y,x) + d(x,p)] + K^3 + 2K + \epsilon \\
&\leq K^2[K + \epsilon + (K+\epsilon)] + K^3 + 2K + \epsilon.
\end{aligned}$$

By decreasing ϵ down to zero, we deduce $d(x,p) \leq K$ and $d(p,x) \leq 3K^3 + 2K$ for any $p \in \pi_\gamma(x)$. ☐

Proof of Lemma 2.2.4. We first show $\mathrm{diam}(\pi_\gamma(w)) \leq 3K^3 + 3K$ for any $w \in X$. If $w \notin \mathcal{N}_K(\gamma)$ then $\mathrm{diam}(\pi_\gamma(w)) < K$ by K-BGIP, and if $w \in \mathcal{N}_K(\gamma)$ then for any $w', w'' \in \pi_\gamma(w)$ we have $d(w', w'') \leq d(w', w) + d(w, w'') \leq K + (3K^3 + 2K)$ by Lemma 9.4.2.

Let us now prove the lemma. If one of $[x,y]$ and $[y,x]$ is disjoint from $\mathcal{N}_K(\gamma)$, then $\mathrm{diam}(\pi_\gamma(\{x,y\})) < K$ by the BGIP. If not, we take $z \in [x,y] \cap \overline{\mathcal{N}_K(\gamma)}$ and $z' \in [y,x] \cap \overline{\mathcal{N}_K(\gamma)}$ such that $[x,z), [y,z')$ are disjoint from $\overline{\mathcal{N}_K(\gamma)}$. In other words, we take z, z' to be the 'leftmost' ones among the candidates.

Then for any $q' \in \pi_\gamma(z)$ and $q \in \pi_\gamma(y)$, we have

$$\begin{aligned}
d(q', q) &\leq d(q', z) + d(z, y) + d(y, q) \\
&\leq 3K^3 + 2K + d(z,y) + d(y, \pi_\gamma(z')) \\
&\leq 3K^3 + 2K + d(z,y) + d(y, z') + d(z', \pi_\gamma(z')) \\
&\leq 3K^3 + 3K + \epsilon.
\end{aligned}$$

Moreover, we have $\mathrm{diam}(\pi_\gamma([x,z])) \leq 3K^3 + 3K$ since either $x = z \in \mathcal{N}_K(\gamma)$ or $[x,z]$ is disjoint from $\mathcal{N}_K(\gamma)$. Hence, we have $d(p,q) \leq 6K^3 + 6K + \epsilon$ for any $p \in \pi_\gamma(x)$ and $q \in \pi_\gamma(y)$.

By symmetry, we also have $d(q,p) \leq 6K^3 + 6K + \epsilon$ for such pair. Finally, we know that $\mathrm{diam}(\pi_\gamma(x)) \leq 3K^3 + 3K$ and $\mathrm{diam}(\pi_\gamma(y)) \leq 3K^3 + 3K$. Combining these, we conclude that $\mathrm{diam}(\pi_\gamma(x) \cup \pi_\gamma(y)) \leq 12K^3 + 12K + \epsilon$. ☐

Corollary 9.4.3 (Continuity of projections II)**.** *Let X be a geodesic space. For each $K > 1$ there exists a constant $K' = K'(K)$ that satisfies the following property.*

Let γ be a K-BGIP axis, $A \subseteq X$ be a connected set and $a \in \mathbb{R}$. If $\gamma^{-1}\pi_\gamma(A)$ is contained in the union of $I_1 := (-\infty, a]$ and $I_2 := [a + K', +\infty)$ then it is contained in either I_1 or I_2.

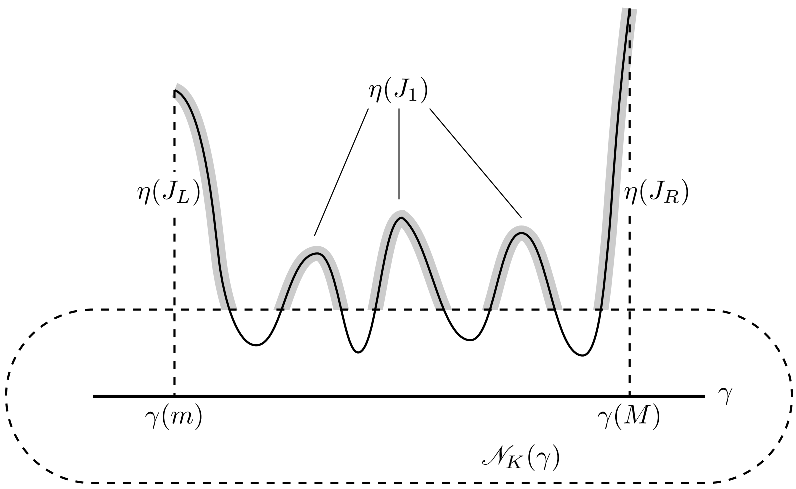

Figure 9.1: Schematics for the proof of Lemma 2.2.5. The projection of $\eta(J')$ onto γ is small for each component J' of $J \setminus J_0$.

Proof of Lemma 2.2.5. Let $K_0 = K'(K)$ be as in Corollary 9.4.3. Let also $J_0 = \{s \in J : \eta(s) \notin \overline{\mathscr{N}_K(\gamma)}\}$, which is open since geodesics are continuous with respect to the d^{sym}-topology on X.

For each component J' of J_0, $\eta\left(\bar{J}'\right)$ is disjoint from $\mathscr{N}_K(\gamma)$ so we have $\operatorname{diam}\left(\pi_\gamma \eta\left(\bar{J}'\right)\right) \le K$. In particular, the assumption $\operatorname{diam}(\pi_\gamma(\eta)) > K$ forces that J_0 has more than 1 component; hence $J \setminus J_0$ is nonempty. We now let

$$A := \inf J \setminus J_0, \quad B := \sup J \setminus J_0$$

and claim that $\gamma([m, M] \cap I)$ and $\eta([A, B] \cap J)$ are close to each other.

First observe that each component of J_0, except the leftmost and the rightmost ones, are shorter than a uniform bound. For such a component $J' = (\alpha, \beta)$, we have $\eta(\alpha), \eta(\beta) \in \partial\mathscr{N}_K(\gamma)$ and $\operatorname{diam}\left(\pi_\gamma \eta\left([\alpha, \beta]\right)\right) < K$. This implies that

$$\begin{aligned}
|\beta - \alpha| &= d(\eta(\alpha), \eta(\beta)) \\
&\le d(\eta(\alpha), \pi_\gamma\eta(\alpha)) + \operatorname{diam}\left(\pi_\gamma\eta(\alpha) \cup \pi_\gamma\eta(\beta)\right) + d(\pi_\gamma\eta(\beta), \eta(\beta)) \\
&\le K + K + [3K^3 + 2K] =: K_1.
\end{aligned}$$

Now let $s \in J$ be such that $A \le s \le B$. By its construction, s either belongs to $J \setminus J_0$ or a component $J' = (\alpha, \beta)$ of J_0 such that $\alpha, \beta \in J \setminus J_0$. In the former case, we have $d^{sym}(\eta(s), \pi_\gamma\eta(s)) \le 3K^3 + 3K$ by Lemma 9.4.2. In the latter case, for any $p \in \pi_\gamma\eta(\beta)$ we have

$$\begin{aligned}
d(\eta(s), p) &\le d(\eta(s), \eta(\beta)) + d(\eta(\beta), \pi_\gamma\eta(\beta)) \\
&\le K_1 + K, \\
d(p, \eta(s)) &\le \operatorname{diam}\left(\pi_\gamma(J')\right) + d(\pi_\gamma\eta(\alpha), \eta(\alpha)) + d(\eta(\alpha), \eta(s)) \\
&\le K + [3K^3 + 2K] + K_1.
\end{aligned}$$

Since $\pi_\gamma\eta(\beta) \subseteq \gamma([m, M] \cap I)$, this establishes one direction.

For the other direction, let us take $t \in I \cap [m, M]$. Let $J_L := J \cap (-\infty, A)$, $J_R := J \cap (B, +\infty)$ and $J_1 := J \cap [A, B]$. Then we have

$$\gamma^{-1}\pi_\gamma(\eta) \subseteq \gamma^{-1}\pi_\gamma(\eta(J_L)) \ \cup\ \gamma^{-1}\pi_\gamma(\eta(J_1)) \ \cup\ \gamma^{-1}\pi_\gamma(\eta(J_R)).$$

Also note that $\gamma^{-1}\pi_\gamma(\eta(J_1))$ is a K_0-connected set by Corollary 9.4.3, and that $\gamma^{-1}\pi_\gamma(\eta(J_L)), \gamma^{-1}\pi_\gamma(\eta(J_R))$ have diameters bounded by $2K^2$. This implies that there exists $t_0 \in I$, $s_0 \in J_1$ such that $\gamma(t_0) \in \pi_\gamma(\eta(s_0))$ and $|t - t_0| \le K_0 + 2K^2$.

If $s_0 \in J \setminus J_0$, then $d^{sym}(\gamma(t_0), \eta(s_0)) < 3K^3 + 3K$ by Lemma 9.4.2; since $\gamma(t)$ and $\gamma(t_0)$ are close to each other, we are done in this case. If $s_0 \in J_0$, it belongs to a component $J' = (\alpha, \beta)$ of J_0 that is not the leftmost or the rightmost one. We then have $\beta \in J \setminus J_0$ and $d^{sym}(\gamma(t_0), \pi_\gamma(\eta(\beta)) \le 2 \operatorname{diam} \pi_\gamma(J') \le 2K$. By replacing s_0 with β and t_0 with an element of $\pi_\gamma(\eta(\beta))$, we similarly deduce the conclusion. □

Proof of Lemma 2.2.6. Let $\gamma : I \to X$ be a K-BGIP axis, $\gamma' = \gamma|_{I'} : I' \to X$ be its subsegment and $\eta : J \to X$ be a geodesic. Let also $I_L := \{x \in I : x < I'\}$, $I_R := \{x \in I : x > I'\}$. Let $K_1 = K'(K)$ be as in Lemma 2.2.5, $K_2 = K'(K)$ be as in Lemma 2.2.4, $R = 3K(K_1 + K_2 + K)$, $R_1 = K_1 + 2(R+1)$ and $K' = KR_1 + K^2$.

Let $z \in \eta$. We first claim that if $\gamma^{-1}(\pi_\gamma(z)) \cap I_R \ne \emptyset$, then $\gamma^{-1}\pi_{\gamma'}(z) \subseteq [\sup I' - R, \sup I']$. If not, then there exists $w \in \pi_{\gamma'}(z)$ such that $\gamma^{-1}(w)$ intersects $(-\infty, \sup I' - R)$. Then $\operatorname{diam}(\pi_\gamma(z) \cup w) \ge R/K - K > K + 1$ so $[z, w]$ passes through $\mathscr{N}_{K_1}(\gamma(\sup I'))$ by Lemma 2.2.5. Let $p \in [z, w]$ be that intersection point. Then

$$\begin{aligned} d(z, \gamma(\sup I' - \epsilon)) &\le d(z, \gamma(\sup I')) + K\epsilon + K \\ &\le d(z, p) + d(p, \gamma(\sup I')) + K\epsilon + K \\ &\le d(z, w) - d(p, w) + d(p, \gamma(\sup I')) + K\epsilon + K \\ &\le d(z, w) - d(\gamma(\sup I'), w) + d(\gamma(\sup I'), p) + d(p, \gamma(\sup I')) + K\epsilon + K \\ &\le d(z, w) + K_1 + K\epsilon + K - (R/K - K) < d(z, w) \end{aligned}$$

for sufficiently small $\epsilon > 0$, which is a contradiction.

By a similar reason, $\gamma^{-1}(\pi_\gamma(z)) \cap I_L \ne \emptyset$ implies $\pi_{\gamma'}(z) \subseteq \gamma([\inf I', \inf I' + R])$. Finally, if $\gamma^{-1}(\pi_\gamma(z)) \cap I' \ne \emptyset$ then $\pi_{\gamma'}(z) = \pi_\gamma(z) \cap \gamma'$.

Let us now suppose that the diameter of $\pi_{\gamma'}(\eta)$ is greater than K'. Without loss of generality, let $x, y \in \eta$ and $s' \in \gamma^{-1}\pi_{\gamma'}(x)$, $t' \in \gamma^{-1}\pi_{\gamma'}(y)$ be such that $t' - s' > K'/K - K = R_1$. We then pick s to be s' if $s' \in \gamma^{-1}\pi_\gamma(x)$ and an arbitrary element of $\gamma^{-1}\pi_\gamma(x)$ if not. Similarly we take $t = t'$ or an element of $\gamma^{-1}\pi_\gamma(y)$.

We claim that $s \le s' + R + 1$. If not, we have either $s \in I_R$ or $s' \le s - R - 1 \le \sup I' - R - 1$. In the former case we have $\sup I' - R \le s' < t' \le \sup I'$ and $t' - s' \le R < R_1$, a contradiction. In the latter case, the previous observation tells us that $\gamma^{-1}(\pi_\gamma(x)) \cap I_R = \emptyset$. This forces one of the following cases:

- $\gamma^{-1}(\pi_\gamma(x)) \cap I' \ne \emptyset$ holds, in which case $\pi_{\gamma'}(x) = \pi_\gamma \cap \gamma'(x)$ and $|s - s'| \le K \operatorname{diam}(\pi_\gamma(x)) + K^2 \le R$ hold; or,
- $\gamma^{-1}(\pi_\gamma(x)) < I'$ and $s \le s'$; in either case we have a contradiction.

By a similar reason, we also deduce $t \ge t' - R - 1$. In conclusion, we have

$$t - s \ge \min(t, t') - \max(s, s') \ge t' - s' - 2(R+1) \ge R_1 - 2(R+1) \ge K_1 \tag{9.4.1}$$

and $\mathscr{N}_K(\gamma(s^*)) \cap \eta \ne \emptyset$ for all $s \le s^* \le t$ by K-BGIP of γ. Also, Inequality 9.4.1 implies that $s^* \in [\min(t, t'), \max(s, s')]$ exists, which clearly belongs to I'. This establishes K'-BGIP of γ'.

We now investigate the second assertion. Let

$$K_3 := 2K^2(10K^3 + K_1 + K_2) + K_2.$$

As before, let $\gamma : I \to X$ be a K-BGIP axis and A be a K-bi-quasigeodesic that is within Hausdorff distance K from γ. For $x \in X$, we claim that $\pi_\gamma(x) \cup \pi_A(x)$ is bounded. To see this, let $z \in \pi_\gamma(x)$ and $z' \in \pi_A(x)$. Since γ and A are within Hausdorff distance K, there exist $w \in \gamma$, $w' \in A$ such that $d^{sym}(w, z'), d^{sym}(w', z) \leq K$. Then for any $w^* \in \pi_\gamma(z')$ we have

$$\begin{aligned}
\operatorname{diam}(z' \cup w^*) &\leq d(z', w^*) + d(w^*, z') \\
&\leq d(z', w) + d(w^*, w) + d(w, z') \\
&\leq d^{sym}(w, z') + K^2 d(w, w^*) + K^3 + K \\
&\leq d^{sym}(w, z') + K^2[d(w, z') + d(z', w^*)] + K^3 + K \\
&\leq (K^2 + 1) d^{sym}(w, z') + K^3 + K \leq 2K^3 + 2K.
\end{aligned}$$

Now, if $d(z, z') \geq 2K^3 + 3K + K_1 + K_2$, then

$$\operatorname{diam}(\pi_\gamma([x, z'])) \geq \operatorname{diam}(z \cup \pi_\gamma(z')) \geq \operatorname{diam}(z \cup z') - \operatorname{diam}(\pi_\gamma(z') \cup z) \geq K$$

and $[x, z']$ passes through $\mathscr{N}_{K_1}(z)$ by K-BGIP of γ. Let $p \in [x, z']$ be a point in the intersection. This implies that

$$\begin{aligned}
d(x, w') &\leq d(x, p) + d(p, w') \\
&\leq d(x, z') - d(p, z') + d(p, z) + d(z, w') \\
&\leq d(x, z') - [d(z, z') - d(z, p)] + d(p, z) + d(z, w') \\
&\leq d(x, z') - (2K^3 + 3K + K_1 + K_2) + K_1 + K < d(x, z'),
\end{aligned}$$

which contradicts the fact that $z' \in \pi_A(x)$. Hence, we conclude that $d(z, z') < 2K^3 + 3K + K_1 + K_2$ and $d(z, w) \leq 2K^3 + 4K + K_1 + K_2$. Since γ is a K-bi-quasigeodesic, we have

$$\begin{aligned}
d(w, z) &\leq K^2(2K^3 + 4K + K_1 + K_2) + K^3 + K, \\
d(z', z) &\leq K^2(2K^3 + 4K + K_1 + K_2) + K^3 + 2K \\
&\leq K^2(10K^3 + K_1 + K_2).
\end{aligned}$$

In short, we have $d^{sym}(z, z') \leq K_3 - K_2$. Since $\operatorname{diam}(\pi_\gamma(x)) \leq K_2$ by Lemma 2.2.4, we conclude that $\operatorname{diam}(\pi_\gamma(x) \cup \pi_A(x)) \leq K_3$.

Now suppose $\operatorname{diam}(\pi_A([x, y])) > 2K_3 + K$. By the previous argument, we deduce that $\operatorname{diam}(\pi_\gamma([x, y])) > K$ and $[x, y]$ passes through $\mathscr{N}_K(\gamma) \subseteq \mathscr{N}_{2K}(A)$. Hence, A has $(2K_3 + 2K)$-BGIP. ☐

From the previous proof we obtain the following corollary.

Corollary 9.4.4 (BGIP is hereditary II)**.** *For each $K > 1$ there exists a constant $K' = K'(K)$ such that the following hold. Let $y \in X$, $\gamma : I \to X$ be a K-BGIP and $\gamma' : I' \to X$ be a subsegment of γ defined on $I' \subseteq I$. Then the diameters of $\pi_{\gamma'}(y) \cup \pi_{\gamma'}(\pi_\gamma(y))$ and $\gamma^{-1}\pi_{\gamma'}(y) \cup \pi_{I'}\gamma^{-1}\pi_{\gamma'}(y)$ are both smaller than K'.*

Proof of Lemma 2.2.7. Let $K_1 = K'(K)$ be as in Lemma 2.2.5 and $K_2 = K'(K)$ be as in Lemma 2.2.4. We claim that $K' = K(2K + 3K_1 + 1)$ works.

Suppose first that $a_1, a_3 \in [a_2 + K', +\infty)$. Let $a := \min\{a_1, a_3\}$. We then have $a \in [a_2, a_1]$, $\gamma(a_i) \in \pi_\gamma \eta(\alpha_i)$ and

$$\operatorname{diam}(\pi_\gamma \eta([\alpha_1, \alpha_2])) \geq \operatorname{diam}(\pi_\gamma \eta(\alpha_2) \cup \pi_\gamma \eta(\alpha_1)) > \frac{1}{K}|a_1 - a_2| - K > K + 1. \tag{9.4.2}$$

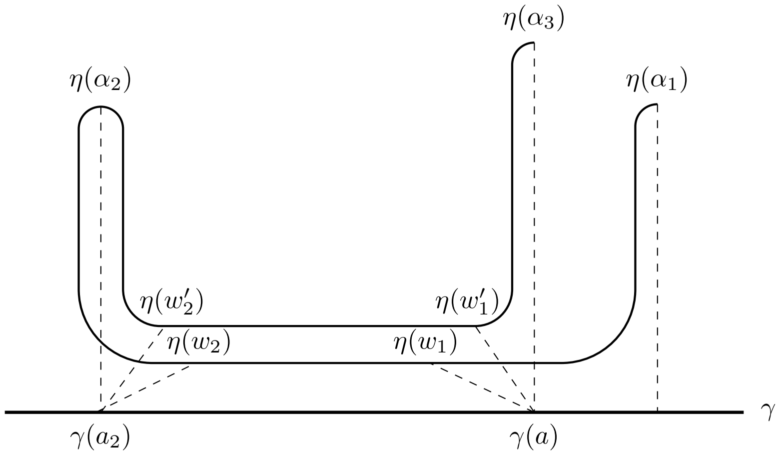

Figure 9.2: Schematics for the proof of Lemma 2.2.7.

Hence, by Lemma 2.2.5, there exists $w_1, w_2 \in [\alpha_1, \alpha_2]$ such that

$$d^{sym}(\eta(w_1), \gamma(a)) < K_1, \quad d^{sym}(\eta(w_2), \gamma(a_2)) < K_1.$$

Similarly, we have $w_1', w_2' \in [\alpha_2, \alpha_3]$ such that

$$d^{sym}(\eta(w_1'), \gamma(a)) < K_1, \quad d^{sym}(\eta(w_2'), \gamma(a_2)) < K_1.$$

Meanwhile, Inequality 9.4.2 also shows that $\mathrm{diam}(\pi_\gamma([\eta(\alpha_1), \gamma(a_2)]))$ is larger than $K+1$. Since $a_2 \leq a \leq a_1$, Lemma 2.2.5 implies that $[\eta(\alpha_1), \gamma(a_2)]$ passes through $\mathscr{N}_{K_1}(\gamma(a))$. Let p be the intersection point and note that

$$\begin{aligned}
d(\eta(\alpha_1), \eta(\alpha_2)) &\geq d(\eta(\alpha_1), \eta(w_2)) \\
&\geq d(\eta(\alpha_1), \gamma(a_2)) - d(\eta(w_2), \gamma(a_2)) \\
&= d(\eta(\alpha_1), p) + d(p, \gamma(a_2)) - d(\eta(w_2), \gamma(a_2)) \\
&\geq [d(\eta(\alpha_1), \gamma(a)) - d(p, \gamma(a))] + [d(\gamma(a), \gamma(a_2)) - d(\gamma(a), p)] - 2K_1 \\
&\geq d(\eta(\alpha_1), \gamma(a)) + \left[\frac{1}{K}|a - a_2| - K\right] - d^{sym}(p, \gamma(a)) - 2K_1 \\
&\geq d(\eta(\alpha_1), \gamma(a)) + \frac{K'}{K} - K - 3K_1.
\end{aligned}$$

By a similar reason, $[\gamma(a_2), \eta(\alpha_3)]$ passes through $\mathscr{N}_{K_1}(\gamma(a))$ and we can deduce

$$d(\eta(\alpha_2), \eta(\alpha_3)) \geq d(\gamma(a), \eta(\alpha_3)) + \frac{K'}{K} - K - 3K_1.$$

Since $\eta(\alpha_1)$, $\eta(\alpha_2)$ and $\eta(\alpha_3)$ are aligned on the same geodesic η, we deduce

$$\begin{aligned}
d(\eta(\alpha_1), \eta(\alpha_3)) &= d(\eta(\alpha_1), \eta(\alpha_2)) + d(\eta(\alpha_2), \eta(\alpha_3)) \\
&\geq d(\eta(\alpha_1), \gamma(a)) + d(\gamma(a), \eta(\alpha_3)) + 2\left(\frac{K'}{K} - K - 3K_1\right) \\
&\geq d(\eta(\alpha_1), \eta(\alpha_3)) + 2\left(\frac{K'}{K} - K - 3K_1\right).
\end{aligned}$$

Since $K' > K(K+3K_1)$, this gives a contradiction. Similar investigation also prevents $a_1, a_3 \in (-\infty, a_2 - K']$. $\square$

Proof of Lemma 2.2.8. For each $0 \le s_1 \le L_1$, let $t \in I$ be such that $d^{sym}(\eta_1(s_1), \gamma(t)) < K$ and let $s_2 \in [0, L_2]$ be such that $d^{sym}(\eta_2(s_2), \gamma(t)) < K$. Then we have

$$\begin{aligned}
|s_1 - s_2| &= |d(\eta_1(0), \eta_1(s_1)) - d(\eta_2(0), \eta_2(s_2))| \\
&\le d^{sym}(\eta_1(0), \eta_2(0)) + |d(\eta_2(0), \eta_2(s_2)) - d(\eta_2(0), \eta_2(s_2))| + d^{sym}(\eta_2(s_2), \eta_1(s_1)) \\
&\le 4K.
\end{aligned}$$

In particular, this implies that $L_2 \ge L_1 - 4M$. By symmetry, $L_1 \ge L_2 - 4M$ also holds.

Now for $0 \le s_1 \le \min\{L_1, L_2\}$, define t and s_2 as above. Let also $t' \in I$ be such that $d^{sym}(\eta_2(s_1), \gamma(t')) < K$. We then have

$$\begin{aligned}
d(\eta_1(s_1), \eta_2(s_1)) &\le d(\eta_1(s_1), \eta_2(s_2)) + d(\eta_2(s_2), \eta_2(s_1)) \\
&\le 2K + d(\eta_2(s_2), \gamma(t)) + d(\gamma(t), \gamma(t')) + d(\gamma(t'), \eta_2(s_1)) \\
&\le 2K + 2K + K|t - t'| + K \\
&\le 5K + K^2 d(\gamma(t'), \gamma(t)) + K^2 \\
&\le (5K + K^2) + K^2[d(\gamma(t'), \eta_2(s_1)) + d(\eta_2(s_1), \eta_2(s_2)) + d(\eta_2(s_2), \gamma(t))] \\
&\le (5K + K^2 + 2K^3) + K^2 d(\eta_2(s_1), \eta_2(s_2)).
\end{aligned}$$

Since one of $d(\eta_2(s_2), \eta_2(s_1))$ and $d(\eta_2(s_1), \eta_2(s_2))$ is bounded by $|s_1 - s_2| \le 4K$, we conclude that $d(\eta_1(s_1), \eta_2(s_1)) \le 6K + K^2 + 6K^3$. Similar estimate holds for $d(\eta_2(s_1), \eta_1(s_1))$. $\square$

9.5 Limit laws on CAT(0) spaces and Outer space

Thanks to the proofs of Lemma 2.2.5, 2.2.6 and Lemma 2.2.7 in the language of BGIP and asymmetric metrics, the concatenation lemmata in Section 3.1 can be immediately brought to CAT(0) spaces and Outer space. We end this dissertation by considering the following general theorems.

Convention 9.5.1. *We assume the following:*

- *(X, d) is a (possibly asymmetric) geodesic metric space;*
- *G is a countable group of isometries of X, and*
- *G contains two independent isometries that satisfy the bounded geodesic image property (BGIP).*

We fix a reference point $o \in X$. μ denotes a non-elementary *discrete probability measure on G, and $\check{\mu}$ denotes its reflected version $\check{\mu}(g) := \mu(g^{-1})$. $\omega = (\omega_n)_{n=1}^{\infty}$ denotes the random walk generated by μ.*

Remark 9.5.2. *The setting as in Convention 9.5.1 includes the following situations:*

1. *(X, d) is a geodesic Gromov hyperbolic space and G contains two independent loxodromics, e.g. (X, d) is the curve complex of a finite-type hyperbolic surface and G is the corresponding mapping class group, or*
2. *(X, d) is the complex of free factors of the free group of rank $N \ge 3$ and G is the outer automorphism group $\mathrm{Out}(F_N)$;*
3. *X is Teichmüller space of finite type, G is the corresponding mapping class group, and d is either the Teichmüller metric $d_{\mathcal{T}}$ or the Weil-Petersson metric d_{WP};*

4. X is Culler-Vogtmann Outer space CV_N for $N \geq 2$, G is the outer automorphism group $Out(F_N)$, and d is the (asymmetric) Lipschitz metric d_{CV};
5. (X, d) is the Cayley graph of a braid group modulo its center $B_n/Z(B_n)$ with respect to its Garside generating set, and G is the braid group B_n [CW21];
6. (X, d) is a (not necessarily proper nor finite-dimensional) CAT(0) space and G contains two independent strongly contracting isometries; e.g., G is an irreducible right-angled Artin group and (X, d) is the universal cover of its Salvetti complex.

Theorem H (SLLN)**.** *Let ω be the random walk on G generated by a non-elementary measure μ. Then there exists a constant $\lambda = \lambda(\mu) \in (0, +\infty]$ such that*

$$\lim_n \frac{1}{n} d(o, \omega_n o) = \lim_n \frac{1}{n} \tau(\omega_n) = \lambda \tag{9.5.1}$$

for almost every ω. Moreover, $\lambda(\mu)$ is finite if and only if μ has finite first moment.

We call $\lambda(\mu)$ in Theorem A the *escape rate* of μ.

Theorem I. *Let ω be the random walk on G generated by a non-elementary measure μ. If μ, $\check{\mu}$ has finite first moment, then there exists $K > 0$ such that*

$$\limsup_{n \to \infty} \frac{1}{\log n} |d(o, \omega_n o) - \tau(\omega_n)| \leq K \quad a.s.$$

Theorem J (CLT and LIL)**.** *Let ω be the random walk on G generated by a non-elementary measure μ. If μ has finite second moment, then there exists a Gaussian law with variance $\sigma(\mu)^2$ to which $\frac{1}{\sqrt{n}}(d(o, \omega_n o) - n\lambda)$ and $\frac{1}{\sqrt{n}}(\tau(\omega_n) - n\lambda)$ converge in law. Here, $\sigma(\mu) > 0$ if and only if μ is non-arithmetic. If $\check{\mu}$ has finite second moment also, then we have*

$$\limsup_{n \to \infty} \pm \frac{d(o, \omega_n o) - \lambda n}{\sqrt{2n \log \log n}} = \limsup_{n \to \infty} \pm \frac{\tau(\omega_n) - \lambda n}{\sqrt{2n \log \log n}} = \sigma(\mu) \quad \textit{almost surely.}$$

Conversely, suppose that μ has infinite second moment. Then for any sequence $(c_n)_n$, both $\frac{1}{\sqrt{n}}(d(o, \omega_n o) - c_n)$ and $\frac{1}{\sqrt{n}}(\tau(\omega_n) - c_n)$ do not converge in law.

Theorem K (Genericity of pseudo-Anosovs I)**.** *Let ω be the random walk on G generated by a non-elementary measure μ. Let $\lambda = \lambda(\mu)$ be the escape rate of μ and $0 < L < \lambda$. Then there exists $K > 0$ such that*

$$\mathbb{P}\left(\omega_n \textit{ has BGIP and } \tau(\omega_n) \geq Ln\right) \geq 1 - Ke^{-n/K}$$

holds for all n.

Theorem L (Geodesic tracking)**.** *Let ω be the random walk on G generated by a non-elementary measure μ.*

1. *Suppose that μ, $\check{\mu}$ has finite p-th moment for some $p > 0$. Then for almost every path $\omega = (\omega_n)_n$, there exists a quasigeodesic γ such that*

$$\lim_n \frac{1}{n^{1/2p}} d^{sym}(\omega_n o, \gamma) = 0.$$

2. *Suppose that μ has finite exponential moment. Then there exists $K < \infty$ satisfying the following: for almost every path $\omega = (\omega_n)_n$, there exists a quasigeodesic γ such that*

$$\limsup_n \frac{1}{\log n} d^{sym}(\omega_n o, \gamma) < K.$$

Theorem M (Genericity of pseudo-Anosovs II)**.** *Let G be a finitely generated non-elementary subgroup of $\mathcal{G}$. Then there exists a finite generating set $S \subseteq G$ such that the proportion of non-BGIP elements in the ball $B_S(n)$ decays exponentially as $n \to \infty$.*

Chapter 10. Discussion and further questions

So far, we have developed a systematic approach to random walks on Teichmüller space using the contracting properties of pseudo-Anosov mapping classes. Our theory culminated in Theorem F that pseudo-Anosovs are generic in the Cayley graph of $\mathrm{Mod}(\Sigma)$ for some generating set.

Many results presented here have been partially observed or predicted by other authors. Nevertheless, exponential bounds for the set of pivotal times are so strong that it leads to the optimal limit laws and deviation inequalities at once; this draws a striking contrast with traditional approaches that required more restrictive moment conditions for the same results. As shown in the proof of Theorem F, these exponential bounds can have other ramifications related to the growth of groups and counting problems.

Still, the genericity of pseudo-Anosov mapping classes in $\mathrm{Mod}(\Sigma)$ has not been settled for an arbitrary generating set. While our strategy is powerful, it requires a considerable proportion of the Schottky set in the generating set. One idea to remove this condition is to cleverly use the acylindrical action of the mapping class group on the curve complex or Teichmüller space. Indeed, the theory of Pierre Mathieu and Alessandro Sisto [MS20] suggests that the word metric on an acylindrically hyperbolic group can be probed by the metric of the hyperbolic space. We hope to expand this idea and construct an effective counting method for an arbitrary generating set.

Another promising direction is to complete a QI-invariant random walk theory, i.e., the theory that relies on the intrinsic geometry of the group itself and does not rely on its action on the ambient space. Unfortunately, the strong contracting property is not QI-invariant (whereas a weaker notion called the weak contracting property is indeed QI-invariant), making our theory not QI-invariant. Hence, although some groups possess strongly contracting isometries for word metrics with respect to particular generating sets (such as braid groups), our random walk theory will not apply to an arbitrary word metric on such groups. Hence, an independent approach using a different hyperbolic-like QI-invariant property is desired. Recently, Antoine Goldsborough and Alessandro Sisto initiated this program [GS21]. It would be great if this direction of research shed light on a new geometric aspect of the mapping class group.

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여기까지 걸어오는 과정에서 수많은 친구들과 함께 했습니다. 고등학교 때부터 십여 년간 저의 일거수일투족을 한결같이 지켜봐 준 고기영 씨, 최의현 씨, 신의진 씨께 특별히 감사의 말씀을 드립니다. 학부 시절에 맺은 인연을 지금까지 이어와 준 전철호 씨, 송석영 씨, 김동관 씨, 권민성 씨, 심혜린 씨, 김하정 씨, 권용휘 씨 등 카이스트신문 기자님들께도 감사드립니다. 여러분의 끊임없는 격려와 배려 덕분에 제가 바로 설 수 있었습니다. 한편, 대학원 과정 중 가끔 힘든 순간도 있었지만, 김민규 씨와 진우영 씨의 밤낮없는 해학을 낙으로 삼아 버틸 수 있었습니다. 학교 안팎에서 소소하게 즐거운 시간들을 같이 보내 준 강다현 씨, 김나영 씨, 박지희 씨, 강규민 씨, 김덕현 씨, 김지욱 씨, 김현웅 씨, 박준우 씨, 류자영, 위성군, 정재원 씨께도

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연 구 업 적

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